Examlet 1 Foundations of Advanced Math 2/1/07

1. a) State the definition of an irrational number.

a real number is irrational if it is not rational

Good!

b) Write the truth table for P ⇒ Q.

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

Yes

2. Determine whether the propositionals P ⇒ (Q ∨ R) and (P ⇒ Q) ∨ (P ⇒ R) are equivalent.

P	Q	R	Q ∨ R	* P=>Q∨R	P=>Q	P=>R	* (P=>Q) ∨ (P=>R)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T
				*			*

The two propositions are equivalent since their corresponding columns in the truth table (indicated by *) are equivalent.

Great!

3. Prove that for integers m and n, $m \cdot n$ is odd if and only if both m and n are odd.

Proof: If an integer m is odd, $m = 2p+1$ for some integer p.
If an integer n is odd, $n = 2q+1$ for some integer q.

Then, $m \cdot n = (2p+1)(2q+1) = 4pq + 2p + 2q + 1 = 2(2pq + p + q) + 1$.

That means $m \cdot n = 2z + 1$ where z is an integer equal to $2pq + p + q$. Therefore, when m and n are both odd, $n \cdot m$ is odd.

Now, consider when m is odd but n is even. Then $m = 2p+1$ for some integer p and $n = 2q$ for some integer q. Then $m \cdot n = (2p+1)(2q) = 4pq + 2q = 2(2pq+q)$. That means $m \cdot n = 2z$ where z is an integer equal to $2pq + q$. Therefore, when m is odd and n is even, $m \cdot n$ is even.

The last case is when both m and n are even. Then $m = 2p$ for some integer p and $n = 2q$ for some integer q. Then $m \cdot n = (2p)(2q) = 4pq = 2(2pq)$. This means $m \cdot n = 2z$ where z is an integer equal to $2pq$. Therefore when m and n are both even, $m \cdot n$ is even.

Since we proved an even times an even is even, an even times an odd is even, and an odd times an odd is odd, assuming an integer is either even or odd, $m \cdot n$ is odd iff both ~~m~~ m and n are odd.

4. Prove that $\sqrt{3}$ is irrational.

Assume it is rational:

$$\sqrt{3} = \frac{a}{b} \qquad a \text{ and } b \text{ are integers}$$

$$3 = \frac{a^2}{b^2}$$

$$3b^2 = a^2$$

Since a^2 is divisible by 3, a must be divisible by 3 (this has previously been shown in threven, throdd, and throddodd problems). Thus, a^2 can be written $(3p)^2$ where p is an integer.

$$3b^2 = (3p)^2$$

$$3b^2 = 9p^2$$

$$b^2 = 3p^2$$

This means b^2 (and thus b) is divisible by 3 as well. Since $\frac{a}{b}$ is the most simplified form, a and b cannot have a common factor. So, since we came to an illogical conclusion, our original assumption must be false.

Thus, $\sqrt{3}$ is irrational. □

Excellent!

5. Prove that $\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$.

$n=1$: $\sum_{r=1}^{1} 2^r = 2^{1+1} - 2$

$2 = 2^2 - 2$

$2 = 2$ ← true for the base case

s'pose this is true for $n=k$: $\sum_{r=1}^{k} 2^r = 2^{k+1} - 2$

Let $n = k+1$: $\sum_{r=1}^{k+1} 2^r = \sum_{r=1}^{k} 2^r + 2^{(k+1)}$

by my inductive hypothesis:

$\left(\sum_{r=1}^{k} 2^r\right) + 2^{(k+1)} = \left(2^{k+1} - 2\right) + 2^{(k+1)}$

Therefore:

$\sum_{r=1}^{k+1} 2^r = 2^{(k+1)+1} - 2$

which shows that $\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$ is true for $n = k+1$

Thus, by induction $\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$.

Beautiful!