Examlet 2 **Foundations of Advanced Math** **2/22/08**

1. a) State the definition of the union of two sets.

$x \in A \cup B$ iff $x \in A$ or $x \in B$

intersection

b) Find $\{0,1,3,4\} \cap \{0,2,4\}$

$\{0,4\}$

Excellent

4 is not included in the interval being subtracted

c) Find $(3,5) - (4, \infty)$

$(3,4]$

2. a) State the definition of the Cartesian product of two sets A and B.

$A \times B$ is the set of all ordered pairs such that the first element is an element of A and the second is an element of B.

b) Find $\{a, b\} \times \{1, 2\}$.

$\{(a,1),(a,2),(b,1),(b,2)\}$

Fantastic

$\{a,b\} \times \{\ \}$

c) Find $\{a, b\} \times \varnothing$. $\{(a,?)\ (b,?)\}$

$\boxed{\varnothing}$

no second element → no ordered pairs → nothing in set → $\varnothing$

3. State and prove the triangle inequality.

For any a, b

$$|a+b| \leq |a| + |b|$$

Proof:

This is from a Theorem in the book.

$$ab \leq |ab| \qquad |ab| = |a||b|$$

$$ab \leq |a||b|$$

multiply both sides by 2:

$$2ab \leq 2|a||b|$$

add a^2+b^2 to both sides:

$$a^2 + 2ab + b^2 \leq a^2 + 2|a||b| + b^2$$

From book theorem, $|a|^2 = a^2$

We can make the substitution.

$$(a+b)^2 \leq |a|^2 + 2|a||b| + |b|^2$$

$$(a+b)^2 \leq (|a| + |b|)^2$$

$$\sqrt{(a+b)^2} \leq \sqrt{(|a| + |b|)^2}$$

From book theorem,

$$\sqrt{|x|^2} = |x|$$

so, letting $x = (a+b)$ we aquire $\sqrt{|a+b|^2} = |a+b|$

$$|a+b| \leq |a| + |b|$$

Wonderful!

4. Let $\{ A_i \mid i \in I \}$ be an indexed family of sets, and let B be a set. Show that

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i).$$

Let $x \in B \cap \bigcup_{i \in I} A_i$. Thus, $x \in B$ and $x \in A_i$ for some $i \in I$

Hence, $x \in B \cap A_i$ for some $i \in I$, and so $x \in \bigcup_{i \in I} (B \cap A_i)$

This shows $B \cap \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} (B \cap A_i)$

Let $x \in \bigcup_{i \in I} (B \cap A_i)$. Thus, $x \in B \cap A_i$ for some $i \in I$.

so $x \in B \wedge x \in A_i$ For some $i \in I$. So $x \in B \cap \bigcup_{i \in I} A_i$

This shows $\bigcup_{i \in I} (B \cap A_i) \subseteq B \cap \bigcup_{i \in I} A_i$

Since they are subsets of eachother,

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i) \quad , \square$$

Good.

5. Suppose that $a, b \in \mathbb{R}$. Show that if $a, b > 0$, then $a < b \Leftrightarrow a^2 < b^2$.

$(\Rightarrow)$ We know $a < b$, and since $a > 0$ we can multiply both sides by a to get $a^2 < ab$. On the other hand, since $b > 0$ we can multiply both sides of $a < b$ by b to get $ab < b^2$. But then $a^2 < ab < b^2$, so $a^2 < b^2$.

$(\Leftarrow)_1$ We showed in the last chapter that the contrapositive of a statement is logically equivalent, so instead of proving $a^2 < b^2 \Rightarrow a < b$, we can prove $\neg(a < b) \Rightarrow \neg(a^2 < b^2)$, or $b \leq a \Rightarrow b^2 \leq a^2$. But this can be proved just like the previous part, so we're done.

$(\Leftarrow)_2$ We know $a^2 < b^2$, and want to conclude $a < b$. S'pose, for the sake of obtaining a contradiction, that $a \geq b$. But then as above we could reach $a^2 \geq ab \geq b^2$, or $a^2 \geq b^2$, contradicting our hypothesis. Thus instead of $a \geq b$, we must have $a < b$ as desired.

$(\Leftarrow)_3$ We have $a^2 < b^2$, or $0 < b^2 - a^2$, or $0 < (b-a)(b+a)$. But since a and b are positive, $a + b > 0$, and dividing both sides by it gives us $0 < b - a$ or $a < b$, as desired. $\square$