Exam 3 Calc 2 4/3/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Write out the first three terms in the sequence $\left\{\frac{1}{n^2}\right\}$.

$a_1 = \frac{1}{n^2} = \frac{1}{(1)^2} = \underline{1}$

$a_2 = \frac{1}{n^2} = \frac{1}{(2)^2} = \underline{\frac{1}{4}}$

$a_3 = 1/n^2 = \frac{1}{(3)^2} = \underline{1/9}$

Good!

b) Find the first three partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$S_1 = 1/(1)^2 = \underline{1}$

$S_2 = 1/(1)^2 + 1/(2)^2 = \underline{5/4}$

$S_3 = 1/(1)^2 + 1/(2)^2 + 1/(3)^2 = 1/9 + 5/4 = \underline{49/36}$

2. Find the sum of the series $6 - 2 + \frac{2}{3} - \frac{2}{9} + \ldots$

ratio: $-\frac{1}{3}$

$\left|-\frac{1}{3}\right| < 1$ so converges

Excellent!

Sum of the series: $\frac{a}{1-r}$

$$\frac{6}{1-\left(-\frac{1}{3}\right)} = \frac{6}{\frac{4}{3}} = \frac{\cancel{6}^{3}}{1} \cdot \frac{3}{\cancel{4}_{2}} = \frac{9}{2}$$

3. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' + y' - y = 0$?

$y = e^{rx}$, $y' = re^{rx}$ $y'' = r^2 e^{rx}$

So,

$2r^2 e^{rx} + re^{rx} - e^{rx} = 0$

$e^{rx}(2r^2 + r - 1) = 0$

Very Good!

Since, $e^{rx} \neq 0$, So, $2r^2 + r - 1$ has to be 0

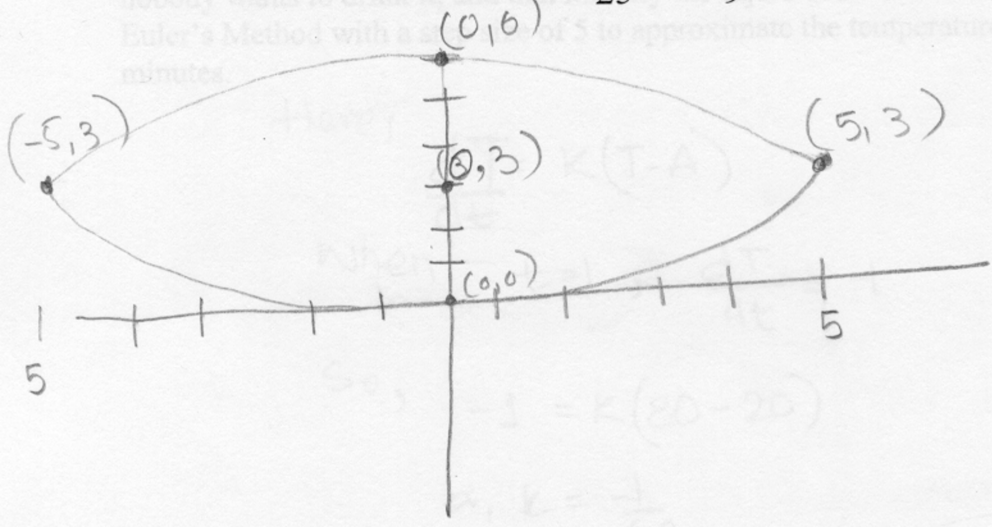
So, $2r^2 + r - 1 = 0$

or, $2r^2 + 2r - r - 1 = 0$ $\therefore \boxed{r = -1 \text{ or, } r = +\tfrac{1}{2}}$

or, $2r(r+1) - 1(r+1) = 0$

or, $(r+1)(2r-1) = 0$ $\therefore r = \{\tfrac{1}{2}, -1\}$

4. Sketch a good graph of the equation $\dfrac{x^2}{25} + \dfrac{(y-3)^2}{9} = 1$.



Yes.

5. Find an equation for the line tangent to the curve with parametric equations $x(t) = t^4 + 1, y(t) = t^3 + t$ at the point where $t = -2$.

$x'(t) = 4t^3$

$y'(t) = 3t^2 + 1$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\frac{dy}{dx} = \frac{3t^2+1}{4t^3}$

$t = -2$ $\quad \frac{3(4)+1}{4(-2)^3} = \frac{13}{-32}$

$\boxed{(y + 10) = \frac{-13}{32}(x - 17)}$

$x(-2) = (-2)^4 + 1 = 17$

$y(-2) = (-2)^3 + -2 = -10$

Well done!

6. Suppose that an 80° cup of very bad soy half-decaf latte is left sitting in a 20° room because nobody wants to drink it, and that initially the liquid cools at a rate of 1° each minute. Use Euler's Method with a step size of 5 to approximate the temperature of the latte after 10 minutes.

$$\frac{dH}{dt} = k(H-A)$$

$$-1 = k(80-20)$$

$$\frac{-1}{60} = k$$

$$\frac{dH}{dt} = \frac{-1}{60}(H-20)$$

Good!

t	H	$\frac{dH}{dt}$	Δt
0	80	-1	-5
5	75	-.917	-4.585
10	70.415		

7. Bunny is a calculus student at Enormous State University, and she has a question. Bunny says "Ohmygod, this is so amazing. I was reading in our Calculus book, like it's the same one you use, right? And there was this example where they, like, showed that the circumference of a circle with radius 1 is 4π instead of 2π! That's so amazing! I thought from Geometry in high school that it was always 2π times the radius, but I didn't know it could be different if you wrote the equation for the circle this parametric way. So, like, I wonder how many other circumferences that circle can have if you take even more math classes?"

Help Bunny by explaining what's going on.

Bunny, be very carefull when you set your limits of integration. I saw that same problem, it was example #4 in 10.2 on parametric calculus. They made that mistake on purpose to show what can happen if you're not careful. They actually found the circumference of 2 circles because they integrated over too large a span of time. In other words they transversed the circle twice. This happens sometimes when you dont notice the equation being $x = \cos 2\theta$ $y = \sin 2\theta$. The "2θ" makes a big diference, because it sends you twice as far on the same limits of integration. So, watch your limits. A tip is watch your calculator graph the shape and note when it completes, only integrate that far.

Wonderful!

8. Suppose that the performance, $P(t)$, of students given a length of time t to learn material, is modeled by the differential equation $\frac{dP}{dt} = k(M - P)$ where M is some positive constant. Find a solution $P(t)$ to this differential equation. What happens to $P(t)$ over the long run?

$$\frac{dP}{dt} = k(M-P)$$

$$dP = dt\, k(M-P)$$

$$\int \frac{1}{M-P}\, dP = \int k\, dt$$

$$-\ln|M-P| = kt + C$$

$$\ln|M-P| = -kt + C_1$$

$$|M-P| = e^{C_1} e^{-kt}$$

A absorbs the e^{C_1} and the ab. bars cause A can be $\pm$ and anything.

Yes!

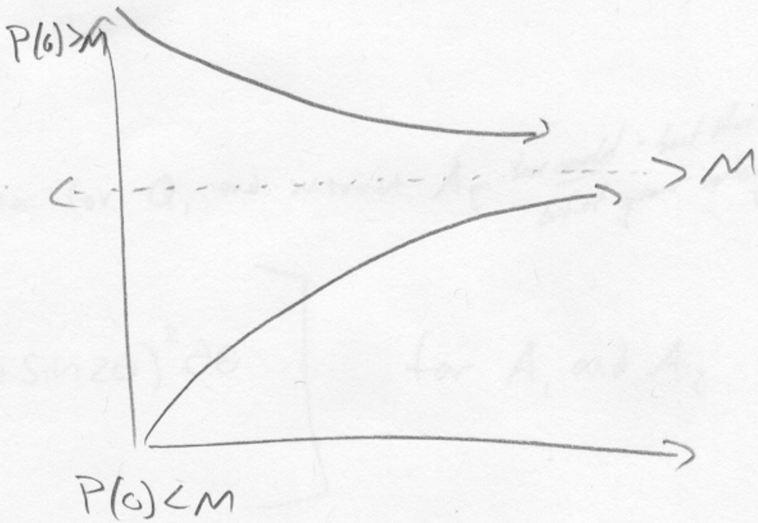
$$M - P = Ae^{-kt}$$

$$-P = Ae^{-kt} - M$$

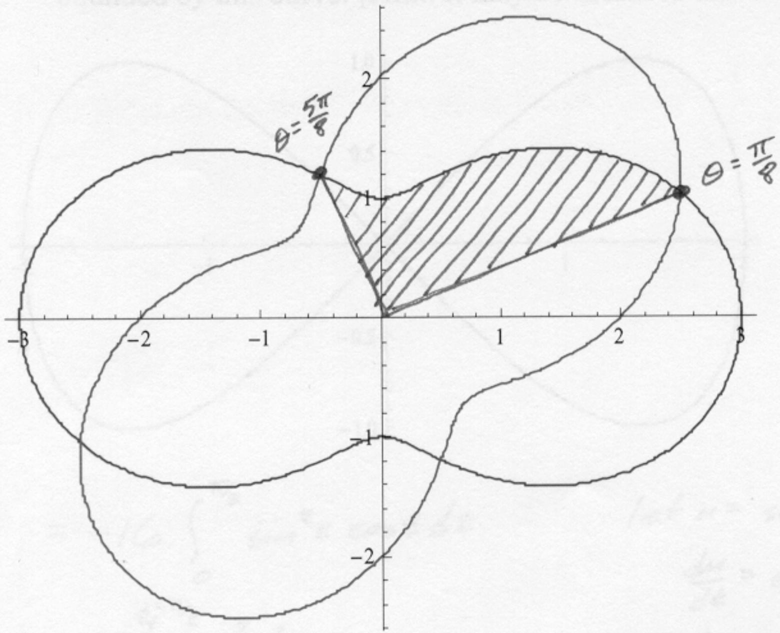
$$P(t) = M - Ae^{-kt}$$

Excellent!

Over the long run $P(t)$ will increase to M, assuming $P(0) < M$ if $P(0) > M$ then $P(t)$ will decrease to M



9. The graphs of $r = 2 + \sin 2\theta$ and $r = 2 + \cos 2\theta$ are shown below. Set up an integral (or integrals) for the area of the region inside both curves.



Where do they cross?

$$2 + \sin 2\theta = 2 + \cos 2\theta$$

$$\sin 2\theta = \cos 2\theta$$

$$\tan 2\theta = 1$$

$$2\theta = \arctan 1 + \pi n$$

$$2\theta = \frac{\pi}{4} + \frac{\pi}{2} n$$

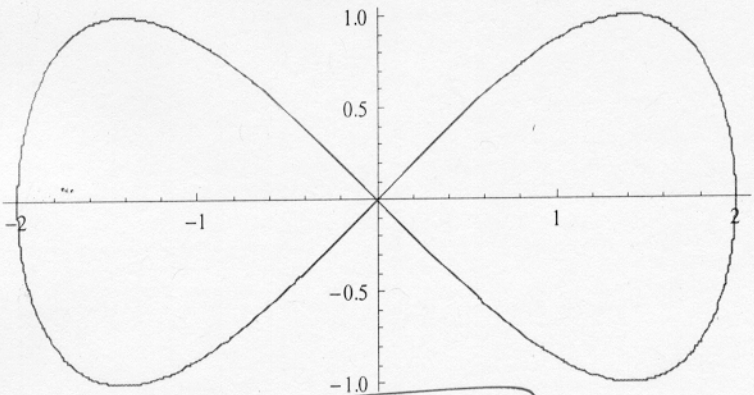
$$\theta = \frac{\pi}{8} + \frac{\pi}{2} n$$

So the first intersection is $\theta = \frac{\pi}{8}$, and the second is $\theta = \frac{5\pi}{8}$.

We'll find the area of the shaded region above and quadruple it:

$$\text{Area} = \left[\frac{1}{2}\int_{\pi/8}^{5\pi/8} (2 + \cos 2\theta)^2 \, d\theta\right] \cdot 4$$

10. The curve with parametric equations $x(t) = 2\cos t$, $y(t) = \sin 2t$ is shown below. Find the area bounded by this curve. [Hint: It may be useful to use the trig identity $\sin 2x = 2\sin x \cos x$.]



Let's find the area in quadrant I and quadruple it. That starts and ends with $y=0$, so

$$0 = \sin 2t$$

$$\arcsin 0 = 2t$$

$$2t = 0 \text{ or } \pi \text{ or} \ldots$$

$$t = 0 \text{ or } \pi/2 \ldots$$

But at $t=0$ we're at $(2,0)$ and at $t=\frac{\pi}{2}$ we're at $(0,0)$, so we're going right-to-left and will get the negative of what we want.

$$\boxed{\text{Area} = \int_\alpha^\beta y(t)\cdot x'(t)\,dt}$$

$$= -4\int_0^{\pi/2} \sin 2t \cdot 2\cos t\,dt$$

$$= -4\int_0^{\pi/2} 2\sin t\cos t \cdot 2\cos t\,dt$$

$$= -16\int_0^{\pi/2} \cos^2 t\,\sin t\,dt$$

let $u = \cos t$

$$\frac{du}{dt} = -\sin t$$

$$\frac{du}{-\sin t} = dt$$

When $t=0$, $u=1$

When $t=\pi/2$, $u=0$

$$= -16\int_1^0 u^2 \cdot \cancel{\sin t} \cdot \frac{du}{\cancel{\sin t}}$$

$$= -16 \cdot \frac{u^3}{3}\Big|_1^0$$

$$= 0 - \frac{-16}{3}$$

$$= \frac{16}{3}$$