Exam 4 Calc 2 4/23/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write a 5th degree MacLaurin polynomial for sin x.

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1!}}$$

$$1 - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Good

$n=0$ $n=1$ $n=2$

2. Give an example of a series which converges, but does not converge absolutely.

$\sum \frac{1}{n}$, Harmonic Series, diverges

$\sum \frac{(-1)^n}{n}$, Alternating Harmonic Series, converges

by the Alternating Series Test.

Great

Therefore it does not converge absolutely.

3. Determine whether the series $\sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{2^n}$ converges or diverges.

Using Ratio test.

$$\lim_{n\to\infty} \left| \frac{\sqrt{n+1+1}}{2^{n+1}} \cdot \frac{2^n}{\sqrt{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{1}{2} \right|$$

Excellent!

$$= \frac{1}{2} \lim_{n\to\infty} \left| \frac{\sqrt{\frac{n}{n}+\frac{2}{n}}}{\sqrt{\frac{n}{n}+\frac{1}{n}}} \right|$$

$$= \frac{1}{2} \cdot 1$$

$$= \frac{1}{2}$$

As, $\frac{1}{2} < 1$, this series converges.

4. Determine the interval of convergence of the MacLaurin polynomial for $f(x) = e^x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| x \cdot \frac{1}{n+1} \right| = \lim_{n\to\infty} \left| x \cdot \frac{1/n}{1+1/n} \right|$$

$$= \lim_{n\to\infty} |x \cdot 0| = 0$$

Hence It converges

& the interval of convergence is $(-\infty, \infty)$ as this series converges for all values of x

Great.

5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

Integral Test: Let $f(x) = \frac{1}{x \ln x}$

$\lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\,dx$ let $u = \ln x$, $du = \frac{1}{x}\,dx$, $dx = x\,du$ $\Longrightarrow \lim_{t\to\infty} \int_2^t \frac{1}{xu} \cdot du\, x$

$= \lim_{t\to\infty} \int_2^t u^{-1}\,du = \lim_{t\to\infty} \Big[\ln(\ln x)\Big]_2^t = \ln(\ln t) - \ln(\ln 2)$

As $t \to \infty$, $\ln(\ln t) \to \infty$ so the definite integral does not equate to a finite number $\therefore$ the series diverges by the integral test

Great

6. Find the 2nd-degree Taylor polynomial for $f(x) = \sqrt[3]{1+x}$ centered at $x = 7$.

Taylor: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

$f(x) = (1+x)^{1/3}$ $\qquad f(7) = (1+7)^{1/3} = 2$

$f'(x) = \frac{1}{3}(1+x)^{-2/3}$ $\qquad f'(7) = \frac{1}{3}(1+7)^{-2/3} = \frac{1}{12}$

$f''(x) = -\frac{2}{9}(1+x)^{-5/3}$ $\qquad f''(7) = -\frac{2}{9}(1+7)^{-5/3} = -\frac{1}{144}$

$a_0 \qquad a_1 \qquad a_2$

$\frac{2}{0!} \qquad \frac{\frac{1}{12}(x-7)}{1!} \qquad \frac{-\frac{1}{144}(x-7)^2}{2!} \Longrightarrow 2 + \frac{x-7}{12} - \frac{(x-7)^2}{288}$

Great

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, I think I'm in trouble. This series stuff is pretty confusing, and all these different tests they've got are pretty crazy. I've got some of it figured out, but what I don't get is are there times when you've gotta use the comparison test instead of the limit comparison?"

Help Biff by answering his questions as clearly as possible.

Biff, you're right that all those tests get confusing. For the Comparison and Limit Comparison Tests, most times one works the other does too, at least if you find the right things to compare to. And when only one of the two works easily, it's usually the Limit Comparison Test rather than the direction you're asking.

But if you really want to know, yes there are some series where C.T. goes well but L.C.T. doesn't. If you think about $\sum \frac{n+1}{n!}$, for instance, the obvious series for L.C.T. would be something like $\sum \frac{1}{n!}$, but the limit doesn't go well so at best you'd have to do something crazier. Another would be $\sum \frac{2+\sin n}{n^2}$, since again the limit involved in comparing to $\sum \frac{2}{n^2}$ is kind of evil.

I suggest you try lots of practice problems from §11.7 and the chapter review to get some of this straight, Biff. The more you do, the more you'll get used to what does and doesn't work out well. Good luck!

8. Is $x = 1$ included in the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$?

$$\sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1}$$ ← this will always be 1.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Use alternating series test!

$f(x) = (2n+1)^{-1}$

$f'(x) - (2n+1)^{-2} \cdot 2$

✓ Signs alternate $(-1)^n$

✓ $\lim_{n\to\infty} \frac{1}{2n+1} = 0$

Excellent!

✓ If $f(x) = \frac{1}{2n+1}$, $f(x) = \frac{-2}{(2n+1)^2}$ so the equation is always decreasing.

By the alternating series test, when $x = 1$ the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges.

9. Use a 4$^{\text{th}}$-degree MacLaurin polynomial for $\cos(x^2)$ to approximate $\int_0^{0.1} \cos(x^2)\,dx$ to the nearest millionth.

I know $\cos x \approx 1 - \dfrac{x^2}{2}$

So $\cos(x^2) \approx 1 - \dfrac{(x^2)^2}{2} = 1 - \dfrac{x^4}{2}$

Then $\displaystyle\int_0^{0.1} \cos(x^2)\,dx \approx \int_0^{0.1} \left(1 - \frac{x^4}{2}\right)dx = \left[x - \frac{x^5}{10}\right]_0^{0.1}$

$= 0.1 - \dfrac{0.1^5}{10} - 0$

$= 0.1 - \dfrac{0.00001}{10} = 0.1 - 0.000001$

$= 0.099999$

10. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with all of its terms positive. What can you say about $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}a_n\right)$?

you know all terms are positive in this series as well.

Try limit comparison since you know $\sum_{n=1}^{\infty} a_n$ is convergent

$$\lim_{n\to\infty} \frac{a_n}{\frac{n+1}{n}a_n} = \lim_{n\to\infty} \frac{1}{\frac{n+1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

which we know is positive and finite, therefore

since $\sum_{n=1}^{\infty} a_n$ converges so does $\sum_{n=1}^{\infty} \frac{n+1}{n} a_n$ by limit comparison test

Excellent!