a) Determine whether the statements $P \Rightarrow Q$ and $\neg P \vee Q$ are equivalent.

Let's write the truth tables for these statements.

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Great!

As you can see, the truth values for the statements $P \Rightarrow Q$ and $\neg P \vee Q$ are the same in all cases, so the statements are logically equivalent. □

b) Determine whether the statements $(P \wedge Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are equivalent.

Let's look at the truth tables.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The statements $(P \wedge Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ do not have the same truth values in some cases, so the statements are not logically equivalent.

Good.

2. a) If a divides b and b divides c, then a divides c.

Well, if $a|b$, then $ax=b$, $x \in \mathbb{Z}$. If $b|c$, then $by=c$, $y \in \mathbb{Z}$.

$$c=by$$
$$c=(ax)y=a(xy)$$

Since $x,y \in \mathbb{Z}$ & two integers multiplied result in an integer, c can be expressed as a times an integer. Therefore, $a|c$. $\square$

Excellent!

b) If $a \equiv_n b$ and $b \equiv_n c$, then $a \equiv_n c$.

Well, if $a \equiv_n b$, $nx=b-a$, $x \in \mathbb{Z}$. If $b \equiv_n c$, $ny=c-b$, $y \in \mathbb{Z}$.

$$nx=b-a$$
$$+\ ny=c-b$$
$$nx+ny=c-a+b-b=c-a$$
$$n(x+y)=c-a$$

Since $c-a$ can be expressed as n times an integer, we can say $n|c-a$. $\therefore$, $a \equiv_n c$. $\square$

Great.

3. $\sqrt[3]{2}$ is irrational.

no common factors.

Well, let's assume $\sqrt[3]{2}$ is rational in hopes of a contradiction; so $\sqrt[3]{2} = \frac{a}{b}$ for two relatively prime integers a and b.

Now, $\sqrt[3]{2} = \frac{a}{b}$

$$2 = \frac{a^3}{b^3}$$

$$2b^3 = a^3$$

From this we see that a^3 is even and since an odd number cubed is odd, a is also even. So we can write a as $2p$ for some integer p.

Now, $2b^3 = (2p)^3$

$$2b^3 = 8p^3$$

$$b^3 = 4p^3$$

$$b^3 = 2(2p^3).$$

From this we see that b^3 is even and thus b is also even. Now we see that a and b have a common factor, namely 2, and are no longer relatively prime so we have reached a contradiction, as hoped.

Good

Thus, $\sqrt[3]{2}$ is irrational. □

Wonderful.

4. Prove that $\forall n \in \mathbb{N}$, $2^n > n$.

Well, for $n=0$ we have $2^0 > 0$, which is true.

Now s'pose it's true for $n=k$, i.e. $2^k > k$. But it's also true that $\forall n \in \mathbb{N}$, $2^n \geq 1$ (see Lemma), ~~so~~ $2^k \geq 1$, and adding gives $2^k + 2^k > k+1$, or $2 \cdot 2^k > k+1$, or $2^{k+1} > k+1$. So since the statement holds for $n=0$, and when it's true for k it's also true for $k+1$, by Mathematical Induction it's true for all $n \in \mathbb{N}$. $\square$

Lemma: $\forall n \in \mathbb{N}$, $2^n \geq 1$.

Proof: Well, when $n=0$, $2^0 \geq 1$.

Now s'pose it holds for $n=k$, so $2^k \geq 1$. Then multiplying by 2 gives $2 \cdot 2^k \geq 2 \cdot 1$ or $2^{k+1} \geq 2$, and $2 \geq 1$, so $2^{k+1} \geq 1$. Thus since it's true for $n=0$ and when it's true for k it's also true for $k+1$, by Mathematical Induction it's true for all $n \in \mathbb{N}$. $\square$

Scratch - Pretend it's not here! :)

Want:

$$2^{k+1} > k+1$$

$$2^k \cdot 2 > k+1$$

$$2^k + 2^k > k+1$$

5. If x is a rational, with $x \neq 0$, and y is irrational, then xy is irrational.

Well lets assume xy is rational x is rational and y is irrational. So $xy = \frac{P}{Q}$ and $x = \frac{r}{s}$ where $P, Q, r, s \in \mathbb{Z}$ and $Q, r, s \neq 0$

so lets substitute $\frac{r}{s} y = \frac{P}{Q}$

$$\frac{ry}{s} = \frac{P}{Q}$$

$$ry = \frac{Ps}{Q}$$

$$y = \frac{Ps}{rQ}$$ and since $P, s, r, Q \in \mathbb{Z}$ Ps is an integer and rQ is also a non zero integer

so y is rational. But we said y was irrational therefore contradicting the assumption we could write xy as a rational number $\therefore$ xy is irrational.

Well done.