Examlet 3 Foundations of Advanced Math 3/26/10

1. Suppose that X is an uncountable set, and A is a subset of X.

   a) Give an example where $X - A$ is uncountable, or explain why this can't happen.

   How about $X = \mathbb{R}$ and $A = \emptyset$, so $X - A = \mathbb{R}$ which is uncountable.

   b) Give an example where $X - A$ is denumerable, or explain why this can't happen.

   How about $X = \mathbb{R}$ and $A =$ the irrationals, so $X - A = \mathbb{Q}$ which is denumerable.

   c) Give an example where $X - A$ is finite, or explain why this can't happen.

   How about $X = \mathbb{R}$ and $A = \mathbb{R}$, so $X - A = \emptyset$, which is finite.

2. Let A be a subset of $\mathbb{R}$ with n elements, where $n \in \mathbb{N}$. Then A is bounded.

Let's use induction. For a set with 0 or 1 elements the result is trivial. For a serious base case consider a set with $n=2$ elements. By Trichotomy one has a larger absolute value (or they're equal), so pick the larger absolute value (or tied value) as our bound.

Now suppose it's true when $n=k$, and consider a set with $k+1$ elements. Remove one element, call it x. The remaining set has k elements, and thus is bounded by hypothesis; call the bound B. Then compare $|x|$ and B.

Case 1: If $|x| \geq B$, then $|x| \geq B > y$ for all other y in the set

Case 2: If $B > |x|$, then $B > y$ for all y in the entire set.

Thus in either case we have a bound ($|x|$ or B, respectively) for our set with $k+1$ elements, so by induction the statement holds. □

3. Complete and defend:

Suppose that f and g are both odd functions from $\mathbb{R}$ to $\mathbb{R}$. Then $f \circ g$ is odd.

Since f is odd, $f(-x) = -f(x)\ \forall x \in \mathbb{R}$.

Since g is odd, $g(-x) = -g(x)\ \forall x \in \mathbb{R}$.

Then $f \circ g(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -f \circ g(x)$

must hold $\forall x \in \mathbb{R}$, so $f \circ g$ is odd. □

4. If $f{:}A \to B$ and $g{:}B \to C$ are surjective functions, then $g \circ f$ is surjective.

Take an arbitrary $c \in C$. To show $g \circ f$ surjective, we must find $a \in A$ such that $g \circ f(a) = c$.

Since g is surjective we know there exists $b \in B$ for which $g(b) = c$. And now since f is surjective we know there exists $a \in A$ for which $f(a) = b$. But then $g \circ f(a) = g(f(a)) = g(b) = c$, so we can always produce the desired a. $\square$

5. Prove, directly from the definitions: If A is a denumerable set, and $b \notin A$, then $A \cup \{b\}$ is denumerable.

Well, since A is denumerable there exists $f: \mathbb{N} \to A$ a bijection. Then make a new function $g: \mathbb{N} \to A \cup \{b\}$ by

$$g(n) = \begin{cases} b & \text{if } n = 0 \\ f(n-1) & \text{if } n > 0 \end{cases}$$

Then g is surjective since for any $x \in A$ we know there's $n \in \mathbb{N}$ for which $f(n) = x$ and thus $n+1 \in \mathbb{N}$ for which $g(n+1) = f(n) = x$. Also g is injective since $g(n_1) = g(n_2)$ implies $n_1 = 0 = n_2$, or $b \in A$ (a contradiction), or $f(n_1 - 1) = f(n_2 - 1)$ which (since f is injective) implies $n_1 - 1 = n_2 - 1$ or just $n_1 = n_2$.

Thus in all cases g is surjective and injective, and hence bijective as desired. □