Exam 1b Calc 2 2/7/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Let $F(x) = \int_0^x \cos\left(t^2\right)dt$. What is $F'(x)$?

$F'(x) = \cos(x^2)$

due to the second part of the fundemental theorem of calculus. (BAM.)

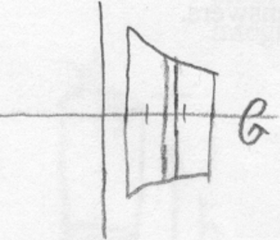
Excellent!

2. Write an integral representing the average value of the function $f(x) = \dfrac{\sin x}{x}$ on the interval $[\pi/2, 3\pi/2]$.

$$A.V. = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\frac{3\pi}{2} - \frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin x}{x}\,dx = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin x}{x}\,dx$$

Good

3. Consider the region bounded between $y = 1/x$, the x-axis, $x = 1$, and $x = 5$. Write an integral for the volume of the solid obtained by rotating this region around the x-axis.



$$\boxed{V = \pi \int_1^5 \left(\frac{1}{x}\right)^2 dx}$$

$V = \pi r^2$
by disks

Great

4. Integrate $\int \frac{x}{\sqrt{1-x^2}}\,dx$.

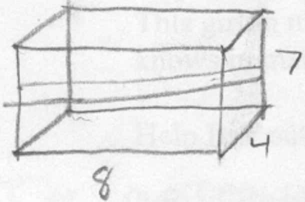
$u = 1-x^2$ $\quad \frac{du}{dx} = -2x \quad -\frac{du}{2} = x\,dx$

$-\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\left(2u^{1/2}+c\right) =$

$-\frac{1}{2}\left(2\sqrt{1-x^2}+c\right) = \boxed{-\sqrt{1-x^2}+c}$

Good.

5. Jon's cat Nemo has a plan to fill his guest bathroom with water and stock it with goldfish. The bathroom is a box 4 feet wide, 8 feet long, and 7 feet deep. Write an integral for the amount of work required to pump all of this water up to the top of the room when Jon cleans up the mess (assume a density of 62.5 lbs/ft^3 for water, and that the goldfish have all mysteriously disappeared before pumping commences).



Area of a slice $8 \times 4 \text{ ft}^2$

Volume of a slice $32 \Delta x \text{ ft}^3$

Weight of a slice $32 \Delta x \text{ ft}^3 (62.5 \text{ lbs}/\text{ft}^3)$

$$\boxed{\text{Work} = \int_0^7 2000\, x \, dx}$$

Excellent!

Assuming all sinks, toilets, etc... have been removed.

6. If a spring has a natural length of 20cm, and 12J of work is required to stretch it from 20cm to 40cm, how much work would be required to stretch it from 20cm to 30cm?

$$12 = \int_0^{0.2} kx\,dx$$

$$12 = \frac{kx^2}{2}\Big|_0^{0.2}$$

$$12 = 0.02k$$

$$k = 600$$

Then the work to stretch from 20cm to 30cm,

$$\text{Work} = \int_0^{0.1} 600x\,dx$$

$$= 300x^2\Big|_0^{0.1}$$

$$= 300 \cdot 0.01$$

$$= 3\text{J}$$

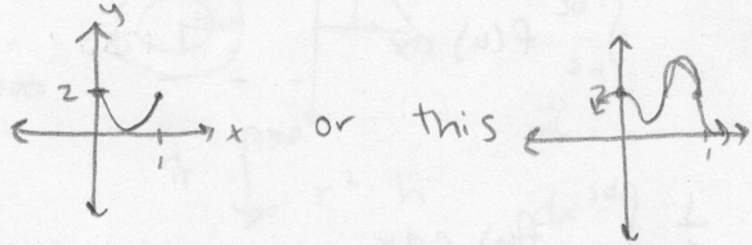
7. Biff is a Calc 2 student at Enormous State University. Biff says "Dang, this Calc stuff is killing me. It's getting all, like, theoretical. There was this one question on our exam last week about the average value stuff, which I'm okay with 'cause I like formulas okay, you know? But so this one was, like, if f(0) is 2, and f(1) is 2, what could the average value of f on the interval from 0 to 1 be? So I said none of the above, since 2 wasn't on the list, but if they don't tell you the formula for what f is I really don't think there's any way to figure it out. This girl in my class said it was all of the above, but that's obviously dumb, since everybody knows in math there's only one right answer."

Help Biff out by explaining clearly what possible average values such functions might have.

$f(0) = 2$
$f(1) = 2$

Dear Biff,

You solution is correct if you were assuming the function was a straight line. What if the function looked like this?



The average values would be completely different. There are endless possibilities since you don't know the function. That is why that girl who must be brilliant, chose all of the above. I would suggest studying with her for your next test. Also remember, the formula for average value is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Great!

8. [Briggs/Cochran, §5.5]

a) A change of variables that can be interpreted geometrically is the scaling $u = cx$, where c is a real number. Prove that

$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,dx.$$

b) Draw a picture to illustrate this change of variables in the case where $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{1}{2}$.

a) We'll evaluate $\int_a^b f(cx)dx$ by a u-substitution. Let $u = cx$,

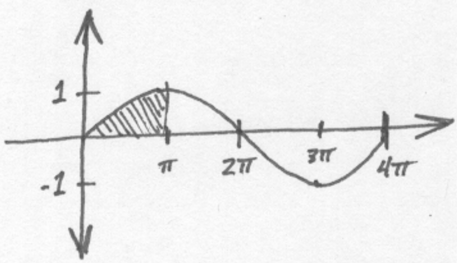
so $\frac{du}{dx} = c$

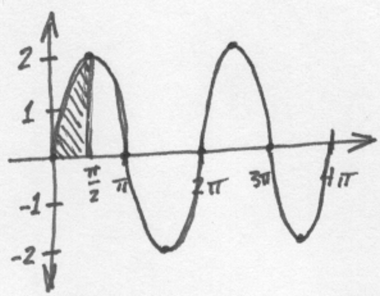
$\frac{du}{c} = dx$

Then $\int_a^b f(cx)dx = \int_{\square}^{\square} f(u)\cdot\frac{du}{c}$. To find the new limits, we plug the old limits $x=a$ and $x=b$ into $u=cx$, giving $u=c(a)$ and $u=c(b)$,

or $\int_a^b f(cx)dx = \int_{ac}^{bc} f(u)\cdot\frac{du}{c} = \frac{1}{c}\int_{ac}^{bc} f(u)du$, as desired. $\square$

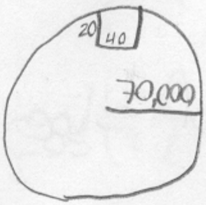
b) $\int_0^{\pi} \sin\left(\frac{x}{2}\right)dx = \frac{1}{1/2}\int_0^{\pi/2} \sin u\, du$

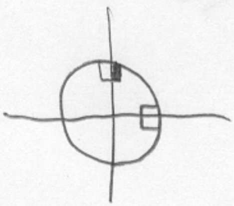




So the shaded region at left is twice as wide, but the shaded region at right is twice as high, and they turn out to have equal areas.

9. In the movie *Star Wars*, the Death Star is a sphere 140,000 meters in diameter. The trench Luke had to fly through was 20 meters deep and 40 meters wide, running around the equator of the sphere. Write an integral for the volume of the trench, that is, the volume of the region between the bottom of the trench and what would have been the surface of the sphere if there were no trench.





$$y^2 = r^2 - x^2$$

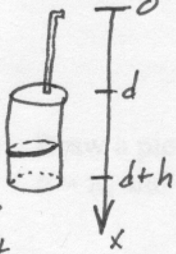
$$x^2 + y^2 = r^2$$

$$y = \pm\sqrt{70{,}000^2 - x^2}$$

$$\underbrace{2\int_0^{20}}_{\text{twice half, one quadrant (symmetry)}} \Big[\underbrace{\pi(70{,}000^2 - x^2)}_{\text{outside washer, curve of sphere}} - \underbrace{\pi(70{,}000-20)^2}_{\text{inside washer, constant distance from the bottom of the trench to the x-axis}} \Big] \underbrace{dx}_{\text{washer thickness}}$$

Excellent!

10. A well is shaped like a cylinder with a vertical axis with height h and radius r, both in feet. If all of the water in the well is to be pumped to a point d feet above the top of the well, write an integral for the total amount of work required.



The distance the slice at the "x" mark on the axis has x units to get to the top of the pipe

Area of a slice = $\pi \cdot r^2 \text{ ft}^2$

Volume of a slice = $\pi r^2 \Delta x \text{ ft}^3$

Weight of a slice = $\pi r^2 \Delta x \text{ ft}^3 \cdot \dfrac{62.5 \text{ lbs}}{\text{ft}^3}$

Work for a slice = $62.5 \pi r^2 \Delta x \text{ lbs} \cdot x \text{ ft}$

Total Work = $\displaystyle\int_d^{d+h} 62.5 \pi r^2 x \, dx \text{ ft} \cdot \text{lbs.}$