Exam 2a Calc 2 3/4/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int x \sin x \, dx$.

Parts

$u = x \qquad v = -\cos x$

$u' = 1 \qquad v' = \sin x$

$= -x\cos x + \int \cos x \, dx = \sin x - x\cos x + C$

Great!

2. **Set up** an integral for the surface area of the solid of revolution obtained by revolving the region below $y = x^3$ between $x = 0$ and $x = 2$ around the x-axis.

$f(x) = x^3 \quad f'(x) = 3x^2$

$$S.A. = 2\pi \int_0^2 f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$\boxed{S.A. = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx}$$

Yes!

3. Evaluate $\int \frac{x-9}{(x+5)(x-2)} dx$.

Partial Fractions

I wish for an A on this test and that

Excellent!

$\int \frac{x-9}{(x+5)(x-2)} dx = \int \frac{A}{x+5} dx + \frac{B}{x-2} dx = 2\int \frac{1}{x+5} dx - \int \frac{1}{x-2} dx$

$u = x+5$, $du = 1$

$x - 9 = A(x-2) + B(x+5)$ $\boxed{2 \ln|x+5| - \ln|x-2| + C}$

if $x = 2$

$-7 = B(7)$

$-1 = B$

if $x = -5$

$-14 = A(-7)$

$2 = A$

4. Find the future value of an income stream of \$2000 per year, for a period of 10 years, if the continuous interest rate is 7%.

$P(t) = 2000$ Future Value $= \int_0^M P(t) e^{r(M-t)}$

$M = 10$

$r = .07$

$F.V. = \int_0^{10} 2000 e^{.07(10-t)} dt$

$= 2000 \int_0^{10} e^{.7 - 0.07t} dt$ $u = .7 - 0.07t$

$\frac{du}{dt} = -0.07$

Well done!

$= \frac{-2000}{0.07} \int_0^{10} e^u \, du$ $dt = \frac{du}{-0.07}$

$= -28571.4 \left[e^{.7 - 0.07t} \right]_0^{10}$

$= -28,571.4 \left[e^{.7 - 0.07(10)} - e^{.7 - 0.07(0)} \right]$

$= -28,571.4 \left[e^0 - e^{.7} \right]$

$= \$28,964.36$

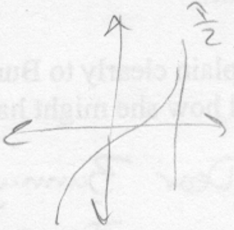
5. Find the value of c for which the function $p(t) = \dfrac{c}{1+x^2}$ is a probability distribution function.

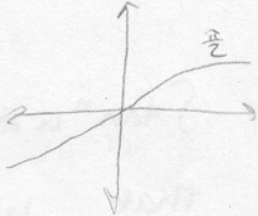
$$\text{p.d.f.} \quad \int_{-\infty}^{\infty} p(t) = 1$$

$$\int_{-\infty}^{0} \frac{c}{1+x^2} + \int_{0}^{\infty} \frac{c}{1+x^2} = 1$$

$$\lim_{b\to-\infty} c\int_{b}^{0} \frac{1}{1+x^2} + \lim_{t\to\infty} c\int_{0}^{t} \frac{1}{1+x^2} = 1$$

Excellent!





$$c\left[\lim_{b\to-\infty} \left[\tan^{-1}x\right]_{b}^{0} + \lim_{t\to\infty} \left[\tan^{-1}x\right]_{0}^{t}\right]$$

$$c\left[\cancel{\tan^{-1}0} - \tan^{-1}b + \tan^{-1}t - \cancel{\tan^{-1}0}\right] = 1$$

$$c\left[-\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}\right] = 1$$

$$\pi c = 1 \qquad \boxed{c = \frac{1}{\pi}}$$

6. Find the length of the curve $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3}$ for $1 \le x \le 2$.

$$f'(x) = \frac{5x^4}{6} + \frac{-3x^{-4}}{10}$$

$$[f'(x)]^2 = \frac{25}{36}x^8 - \frac{30}{60} + \frac{9}{100}x^{-8}$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{25}{36}x^8 - \frac{30}{60} + \frac{9}{100}x^{-8}}\, dx$$

$$= \int_1^2 \sqrt{\frac{25}{36}x^8 + \frac{30}{60} + \frac{9}{100}x^{-8}}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right) dx$$

$$= \left[\frac{1}{6}x^5 - \frac{1}{10}x^{-3}\right]_1^2$$

$$= \left(\frac{2^5}{6} - \frac{1}{10\cdot 2^3}\right) - \left(\frac{1}{6} - \frac{1}{10}\right) = \frac{31}{6} - \frac{1}{80} + \frac{1}{10}$$

$$= \frac{1261}{240}$$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is sooooo impossible. These integrals are crazy, and even once they said we could use the table it's still so confusing. There was this one on the test, and we were supposed to integrate tangent, right? So I found Line 75 on our table, it's the same book as you guys have, and I used it, and so there was 0 in the bottom of the fraction and I marked "Does Not Exist" for the answer, but they said it was wrong. Is that totally unfair or what?"

Explain clearly to Bunny what limitations the Table of Integrals might have that affected her, and how she might have used it better.

line 75 is about $\int \tan^n u \, du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u \, du$

it looks like it an solve all problems of $\tan^n u$ with n can be any number. But that's not true. When $n=1$, $\frac{1}{n-1}$ does not exist. It doesn't mean the integral doesn't exist, but just that this function (line 75) does not apply. If Bunny uses line 12 instead, which is

$\int \tan u \, du = \ln|\sec u| + C$, she'd get the correct answer.

When using the table, it is important to have basic calculus knowledge and know the conditions and limitations.

Excellent!

8. Evaluate $\int \tan\theta \sec^n \theta d\theta$.

Sub $\rightarrow$ $u = \sec\theta$

$\frac{du}{d\theta} = \tan\theta\sec\theta$

$d\theta = \frac{du}{\tan\theta\sec\theta}$

$= \int u^{(n-1)}\, du = \frac{u^n}{n} + C$

Excellent.

$= \frac{\sec^n\theta}{n} + C$

9. Derive Line 30 on the Table of Integrals.

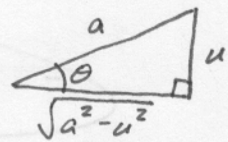
$$\int \sqrt{a^2 - u^2}\, du = \int \sqrt{a^2 - a^2 \sin^2\theta} \cdot a\cos\theta\, d\theta \quad \text{Let } u = a\sin\theta$$

$$= a^2 \int \sqrt{1 - \sin^2\theta} \cdot \cos\theta\, d\theta \qquad \frac{du}{d\theta} = a\cos\theta$$

$$du = a\cos\theta\, d\theta$$

$$= a^2 \int \sqrt{\cos^2\theta}\, \cos\theta\, d\theta \qquad \text{so } \frac{u}{a} = \sin\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$



Line 64

$$= a^2 \cdot \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right) + C$$

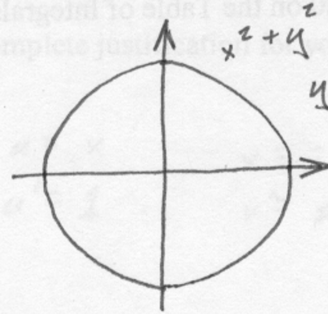
$$= a^2 \cdot \frac{1}{2}\theta + a^2 \cdot \frac{1}{4} \cdot 2\sin\theta\cos\theta + C$$

$$= \frac{a^2}{2} \cdot \sin^{-1}\frac{u}{a} + \frac{a^2}{2} \cdot \frac{u}{a} \cdot \frac{\sqrt{a^2 - u^2}}{a} + C$$

$$= \frac{a^2}{2}\sin^{-1}\frac{u}{a} + \frac{u}{2}\sqrt{a^2 - u^2} + C$$

10. Find the x coordinate of the centroid of the first-quadrant portion of a circle with radius r.

$$\bar{x} = \frac{\int_a^b x \cdot f(x)\,dx}{\int_a^b f(x)\,dx}$$



$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2}$ for top half

$$= \frac{\int_0^r x \cdot \sqrt{r^2 - x^2}\,dx}{\int_0^r \sqrt{r^2 - x^2}\,dx}$$

Let $u = r^2 - x^2$

$\frac{du}{dx} = -2x$

$\frac{du}{-2x} = dx$

$x = 0 \rightarrow u = r^2$

$x = r \rightarrow u = 0$

By #9

$$= \frac{\int_{r^2}^0 \cancel{x} \cdot u^{1/2} \cdot \frac{du}{-2\cancel{x}}}{\left[\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\right]_0^r}$$

$$= \frac{-\frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_{r^2}^{0}}{\left(0 + \frac{r^2}{2} \cdot \frac{\pi}{2}\right) - (0 + 0)}$$

$$= \frac{-\frac{1}{3}\left(0 - (r^2)^{3/2}\right)}{\frac{\pi r^2}{4}}$$

$$= \frac{r^3}{3} \cdot \frac{4}{\pi r^2}$$

$$= \frac{4}{3\pi} \cdot r$$

And that's a little less than $\frac{1}{2} r$, which looks reasonable.