Exam 4a Calc 2 4/22/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give an example of a series which converges, but does not converge absolutely.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}} \rightarrow \text{converges } \underline{\text{A.S.T}}$$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{n^{1/2}} \rightarrow \text{diverges } \frac{\text{P-series}}{P \leq 1}$$

Good!

no Absolute!

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

P series $\sum \frac{1}{n^p} = \sum \frac{1}{n^2}$ P = 2

diverges

$\frac{1}{n^p}$ P series if $p \le 1$ / $p > 1$ in the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ then the series converges

Great!

Since $p = 2 > 1$ the series converges

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ converges or diverges.

Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n+1}} = \lim_{n\to\infty} \frac{2n+1}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{1} = 2$$

2 is a finite # and $\frac{1}{n}$ diverges because it is the harmonic series. So by the Limit Comparison Test $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

Excellent!

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ converges or diverges.

A.S.T.!

✓ $(-1)^n$ makes signs alternate

✓ $\lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0$

✓ If $f(x) = \frac{1}{\sqrt{x+2}} = (x+2)^{-1/2}$,

$f'(x) = -\frac{1}{2}(x+2)^{-3/2} = \frac{-1}{2(x+2)^{3/2}}$

which is negative over positive, so with a negative derivative f decreases.

So we meet the three conditions of the A.S.T. and the series converges.

5. Write the first 3 non-zero terms in the Taylor series for $f(x) = \sin x$ centered at $x = \pi/2$.

Taylor Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$f(x) = \sin x \quad f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x \quad f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x \quad f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x \quad f'''(\frac{\pi}{2}) = 0$

$f^{(4)}(x) = \sin x \quad f^{(4)}(\frac{\pi}{2}) = 1$

repeats

$$\boxed{1 - \frac{1}{2!}(x-\frac{\pi}{2})^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4}$$

Great!

6. Find the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

Rat. Test!

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}$$

$$= 0$$

So since $0<1$ regardless of x, the radius of convergence is ∞.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. Our Calc book is so unfair. There are these questions in it, like where you're supposed to explain why something is true or not, right? Which is stupid, because you know they can't put something like that on a multiple choice exam anyway, right? But our discussion section teacher said the professor was really excited about trying to see if we had conceptual knowledge, whatever that is, so we should pay special attention to this one that asked, like, if $\Sigma\, a_n$ converges, then does $\Sigma\, (a_n + 0.0001)$ converge too. So it seems like 0.0001 isn't big enough to matter, but it feels like maybe a trick question, so I don't know."

Help Bunny by explaining what conclusions could be drawn here and why as clearly as possible.

Bunny, I don't think both of those can converge. Since $\sum a_n$ converges, you know $\lim_{n\to\infty} a_n = 0$, because otherwise $\sum a_n$ would diverge by the Test for Divergence. But then $\lim_{n\to\infty}(a_n + 0.0001) = 0.0001$, so $\sum (a_n + 0.0001)$ diverges by the Test for Divergence.

Basically what it comes down to is that they can't both converge, because you can't have both series consist of terms that go to 0.

8. Find the MacLaurin series for $f(x) = \arctan x$.

As long as $|x| < 1$

I remember: $\sum_{n=0}^{\infty} x^n \overset{*}{=} \frac{1}{1-x}$ because it's geometric

So substitute $(-x^2)$: $\sum_{n=0}^{\infty} (-x^2)^n \overset{*}{=} \frac{1}{1-(-x^2)}$

Rewrite: $\sum_{n=0}^{\infty} (-1)^n x^{2n} \overset{*}{=} \frac{1}{1+x^2}$

Antidifferentiate: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \overset{*}{=} \arctan x + C$

Finally note that $C = 0$ since $\arctan 0 = 0$, and we have

$$\boxed{\arctan x \overset{*}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

9. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges or diverges.

Integral Test (it's decreasing for $n \geq 2$ and always positive)

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx$$

Parts $\quad u = \ln x \qquad v = \frac{x^{-2}}{-2}$

$\qquad\quad u = \frac{1}{x} \qquad v' = x^{-3}$

$$= \lim_{b\to\infty} \left[-\frac{x^{-2}}{2} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^{-2}}{-2}\,dx \right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{2x^2} + \int \frac{1}{2x^3}\,dx \right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{2x^2} + \frac{1}{2} \cdot \frac{x^{-2}}{-2} \right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{2x^2} + \frac{-1}{4x^2} \right]_1^b$$

$$= \lim_{b\to\infty} \left[\left(-\frac{\ln b}{2b^2} + \frac{-1}{4b^2} \right) - \left(\frac{-\ln 1}{2\cdot 1^2} + \frac{-1}{4\cdot 1^2} \right) \right]$$

$$= \lim_{b\to\infty} \frac{-\ln b}{2b^2} + \lim_{b\to\infty} \frac{-1}{4b^2} + 0 + \frac{1}{4}$$

$$\overset{L'H}{=} \lim_{b\to\infty} \frac{-\frac{1}{b}}{4b} + 0 + \frac{1}{4}$$

$$= \lim_{b\to\infty} \frac{-1}{4b^2} + \frac{1}{4}$$

$$= 0 + \frac{1}{4}$$

$$= \frac{1}{4}$$

10. Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$.

Rat. Test!

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1} x^{n+1}}{(n+1)^2}}{\frac{2^n x^n}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{n^2+2n+1} \cdot \frac{n^2}{2^n x^n} \right|$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} |2x|$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{2n+2} |2x|$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2}{2} |2x|$$

$$= |2x|$$

So it converges when $|2x| < 1$, or when $|x| < \frac{1}{2}$, so $\frac{-1}{2} < x < \frac{1}{2}$.

For $x = \frac{1}{2}$ it boils down to $\sum \frac{2^n (\frac{1}{2})^n}{n^2} = \sum \frac{1}{n^2}$, which is a convergent p-series

For $x = \frac{-1}{2}$ it amounts to $\sum \frac{2^n(-\frac{1}{2})^n}{n^2} = \sum \frac{(-1)^n}{n^2}$. Since the absolute values of those terms converge (see above), the series is absolutely convergent and thus also convergent, so...

The interval of convergence is $\left[\frac{-1}{2}, \frac{1}{2}\right]$.