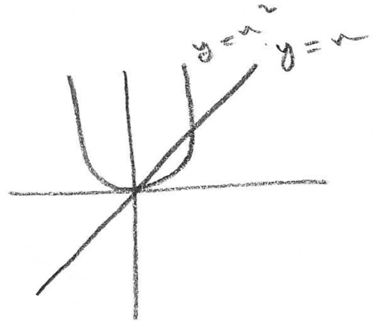
Exam 1 Calc 2 2/3/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an integral for the area of the region bounded between $y = x$ and $y = x^2$.



$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

Good

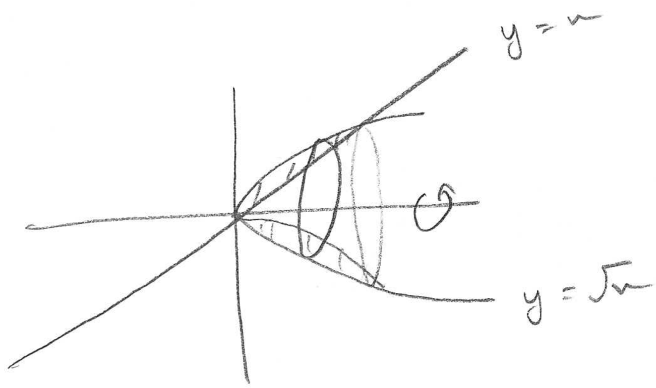
$$A = \int_0^1 (x - x^2)\,dx$$

2. Let $F(x) = \int_0^x \cos\left(t^2\right)dt$. What is $F'(x)$?

$F'(x) = \cos(x^2)$ by the Fundamental Theorom of Calculus, Part 1.

Yes!

3. Set up an integral for the volume of the solid generated when the region bounded between $y = x$ and $y = \sqrt{x}$ is revolved around the x-axis.



$\sqrt{x} - x = 0$

$\sqrt{x}(1 - \sqrt{x}) = 0$

$x = 0, \pm 1$

Great

$$V = \pi \int_0^1 \left[(\sqrt{x})^2 - (x)^2 \right] dx$$

4. If a spring has a natural length of 30cm, and 50N of force is required to hold it stretched to 32cm, how much work would be required to stretch it from 30cm to 32cm?

$W = f \cdot d$ $\quad F = m \cdot a$ $\quad f = kx$ $\quad k$ = constant, x = distance

$f = kx$

$50\text{ N} = k(0.02\text{ m})$

$2500 = k$

$$W = \int_0^{0.02} 2500x \, dx$$

$$\left[1250x^2\right]_0^{0.02}$$

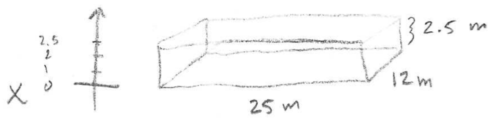
$= 1250(0.02)^2$

$= 1250(0.0004)$

$\boxed{= 0.5 \text{ J of Force}}$

Excellent!

5. A really boring swimming pool is shaped like a box with a base measuring 25m by 12m by 2.5m deep. Set up an integral for the amount of work required to pump all of the water out of the pool when it starts out full.



AREA of a Slice = 25 m × 12 m

VOLUME of a Slice = 300 Δx m^3

Density of a Slice = 300 Δx m^3 · 1000 kg/m^3

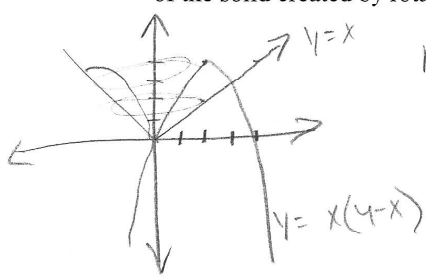
Force of a Slice = 300,000 Δx kg · 9.8 m/s^2

Work of a Slice = 2,940,000 Δx N · x m

$$\boxed{\text{WORK} = \int_0^{2.5} 2{,}940{,}000\, x \, dx}$$

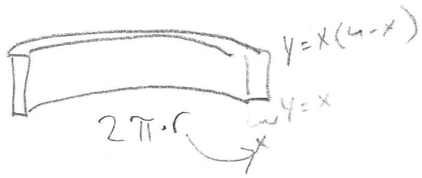
Excellent.

6. Let R be the region bounded between $y = x(4-x)$ and $y = x$. Set up an integral for the volume of the solid created by rotating R around the y-axis.



meet up where x = 3 or 0 and y = 0 or 3

$$\boxed{V = \int_0^3 2\pi x\left(x(4-x) - x\right) dx}$$



Excellent

7. Because of a stunningly negligent editor named Brian, the first printing of the second edition of CliffsQuickReview Calculus was released with a table of integrals that said

$$\int \frac{dx}{a^2+x^2} = \frac{1}{2}\arctan\frac{x}{2} + C.$$

Explain, in simple enough terms that Brian can follow along (Brian claimed to have taken calculus himself), exactly how this formula is or is not acceptable and why.

This formula is not acceptable for all values of a because whenever we integrate a function & get the answer, the answer's derivative should be our initial function. So, here

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)$$

$$\frac{1}{2}\left(\frac{1}{1+(x/2)^2}\right)\cdot\frac{1}{2}$$

$$= \frac{1}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right) = \frac{1}{4}\left(\frac{4}{4+x^2}\right) = \frac{1}{2^2+x^2}$$

This prove that LHS = RHS, if & only if $a=2$

Excellent!

8. Find the exact length of the arc of $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on the interval [1, 3].

$y' = x^3 - \frac{1}{4x^3}$

$\text{Lenth} = \int_1^3 \sqrt{1 + (x^3 - \frac{1}{4x^3})^2}$

$= \int_1^3 \sqrt{1 + x^6 - \frac{1}{2} + \frac{1}{16x^6}}$

$= \int_1^3 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16x^6}}$

$= \int_1^3 x^3 + \frac{1}{4x^3}$

$= \left[\frac{1}{4}x^4 - \frac{1}{8}x^{-2}\right]_1^3$

$= \frac{1}{4} \times 81 - \frac{1}{8} \times \frac{1}{9} - \frac{1}{4} + \frac{1}{8}$

$= \frac{1}{4}(81 - 1) + \frac{1}{8}(1 - \frac{1}{9})$

$= 20 + \frac{1}{8} \times \frac{8}{9}$

$= \frac{180}{9} + \frac{1}{9}$

$= \frac{181}{9}$

Great Job!

9. Derive the integration formula $\int \frac{x}{ax+b}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$.

$$\int \frac{x}{ax+b}\,dx = \int \frac{x}{u}\cdot\frac{1}{a}\,du$$

Let $u = ax+b$

$$\frac{du}{dx} = a$$

$$x = \frac{u-b}{a} \qquad \frac{1}{a}\,du = dx$$

$$= \frac{1}{a}\int \frac{\frac{u-b}{a}}{u}\,du$$

$$= \frac{1}{a^2}\int \frac{u-b}{u}\,du$$

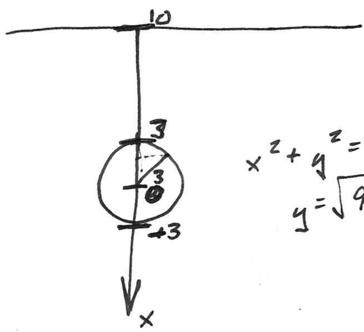
$$= \frac{1}{a^2}\int\left(1 - \frac{b}{u}\right)du$$

$$= \frac{1}{a^2}\cdot\left(u - b\ln|u|\right) + C$$

$$= \frac{1}{a^2}\left(ax+b - b\cdot\ln|ax+b|\right) + C$$

$$= \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + \underbrace{\frac{b}{a^2} + C}_{\text{This is all just another constant.}}$$

10. A water storage tank shaped like a sphere with radius 3m is buried so that the top of the sphere is 7m below ground level. Write an integral for the amount of work required to pump all of the water in this tank up to ground level.



$x^2 + y^2 = 3^2$

$y = \sqrt{9 - x^2}$

Radius of a slice $= \sqrt{9-x^2}\ m$

Area of a slice $= \pi\left(\sqrt{9-x^2}\right)^2\ m^2$

Volume of a slice $= \pi\left(9-x^2\right)\ m^2 \cdot \Delta x\ m$

Mass of a slice $= \pi\left(9-x^2\right)\Delta x\ m^3 \cdot \frac{1000\ kg}{m^3}$

Force for a slice $= 1000\pi\left(9-x^2\right) kg \cdot 9.8\ m/s$

Work for a slice $= 9800\pi\left(9-x^2\right) N \cdot (10-x)$

Total Work $= \int_{-3}^{3} 9800\pi\left(9-x^2\right)(10-x)\,dx\ J$