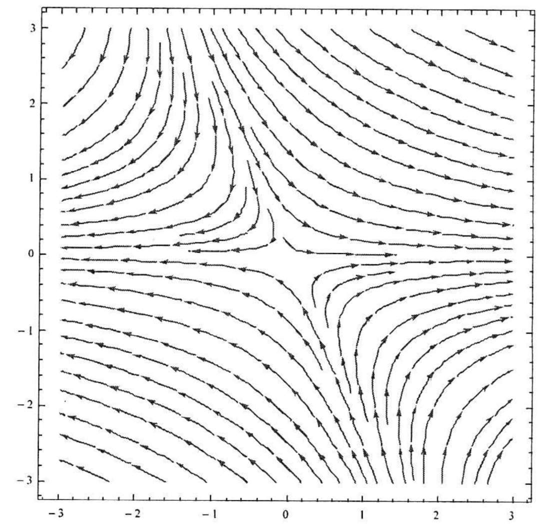
Exam 3 Differential Equations 4/13/12

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. One of the planar systems whose phase plane is shown at right has two real eigenvalues, and the other has purely imaginary eigenvalues. Identify which is which.

saddle

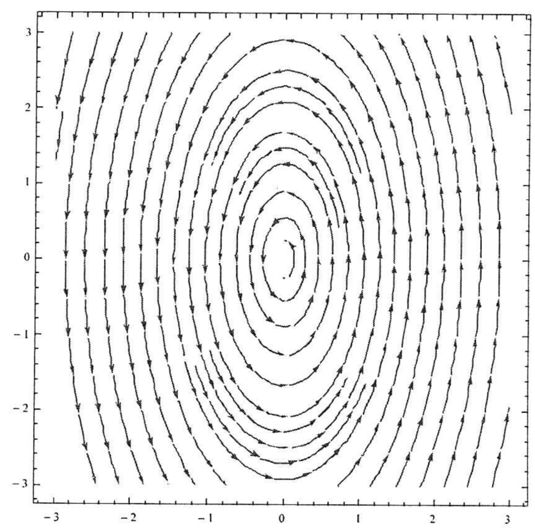
one positive + one negative real eigenvalue



Great

Center

purely imaginary eigenvalues



2. State the Great Theorem of Page 305.

If $\frac{dY}{dt} = AY$ for a 2×2 matrix A and A has a repeated eigenvalue λ then for any initial conditions $\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$

$$Y(t) = e^{\lambda t}\vec{V}_0 + te^{\lambda t}\vec{V}_1 \quad \text{where } \vec{V}_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \text{ and } \vec{V}_1 = (A - \lambda I)\vec{V}_0$$

is a solution.

Excellent!

3. If a planar system of differential equations has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$ and associated eigenvectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (-2,1)$, write a general solution to the system.

$$\hat{Y} = k_1 e^{3t}\begin{pmatrix}1\\0\end{pmatrix} + k_2 e^{t}\begin{pmatrix}-2\\1\end{pmatrix}$$

Yes

4. Find a solution to the initial-value problem $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$, $y(0) = 0$, $y'(0) = 2$.

$y'' + 5y' + 6y = 0$

Let $y = e^{st}$

$y' = se^{st}$

$y'' = s^2 e^{st}$

10

$e^{st}(s^2 + 5s + 6) = 0$

$(s+3)(s+2) = 0$

$s = -3, -2$

$$\boxed{y(t) = -2e^{-3t} + 2e^{-2t}}$$

General Solution: $y(t) = Ae^{-3t} + Be^{-2t}$

Well done

$y(0) = A + B = 0 \Rightarrow A = -B$

$y'(0) = -3A - 2B = 2$

$-3(-B) - 2B = 2$

$3B - 2B = 2$

$B = 2$

$A = -2$

5. Consider the linear system $\frac{d\mathbf{Y}}{dx} = \begin{pmatrix} 0 & 2 \\ -2 & -1 \end{pmatrix}\mathbf{Y}$. Is the origin a spiral source, spiral sink, or center?

$$\det\begin{pmatrix} 0-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = 0$$

$$-\lambda(-1-\lambda) + 4 = 0$$

$$\lambda + \lambda^2 + 4 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-16}}{2}$$

$$= -\frac{1}{2} \pm \frac{i}{2}\sqrt{15}$$

Excellent!

This is a spiral sink since the real component of λ is negative.

6. Consider the linear system $\frac{d\mathbf{Y}}{dx} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{Y}$. Find a general solution to this system.

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-\lambda)(4-\lambda) - (1)(-1) = 8 - 2\lambda - 4\lambda + \lambda^2 + 1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)(\lambda-3) = 0$$

$\lambda = 3 \leftarrow$ repeated eigenvalue

$$\hat{Y}(t) = e^{\lambda t}V_0 + te^{\lambda t}V_1, \quad V_1 = (A - I\lambda)V_0$$

$$\left[\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}$$

$$\hat{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t}\begin{pmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}$$

Excellent!

7. State and prove the Bandicoot Theorem.

For a differential equation $\frac{dY}{dt} = AY$, where λ is an eigenvalue and $\vec{v}$ a corresponding eigenvector of A $Y = e^{\lambda t}\vec{v}$ is a solution.

Proof: $Y = e^{\lambda t}\vec{v}$

$$\frac{dY}{dt} = \lambda e^{\lambda t}\vec{v}$$ ← derivative

$$= \lambda \vec{v} e^{\lambda t}$$

$$= A\vec{v}e^{\lambda t}$$ ← def of eigenvalue/eigenvector

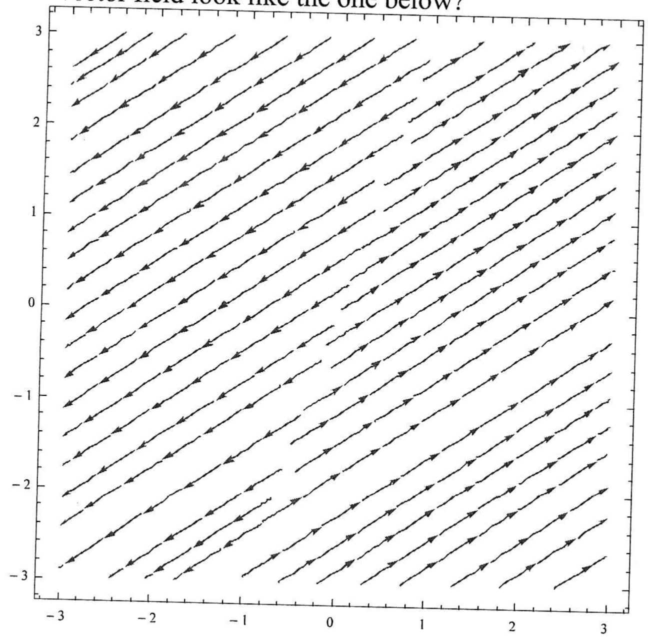
$$= Ae^{\lambda t}\vec{v}$$

$$= AY$$

Since it satisfies the differential equation it is a solution.

Great!

8. Consider the family of linear systems $\dfrac{d\mathbf{Y}}{dx}=\begin{pmatrix}3 & -1\\ 2 & d\end{pmatrix}\mathbf{Y}$. For which values of d will the vector field look like the one below?



This is a phase diagram for when one eigenvalue is 0 so look for when one will be 0.

$(3-\lambda)(d-\lambda)+2=0$

$3d-3\lambda-\lambda d+\lambda^2$

$\lambda^2+(-3-d)\lambda+3d+2=0$

$\underbrace{\lambda(\lambda+(-3-d))+3d+2=0}$

This portion will give $\lambda=0$ if $3d+2=0$

So $3d+2=0$

$3d=-2$

$\boxed{d=\frac{-2}{3} \qquad \frac{dY}{dt}=\begin{pmatrix}3 & -1\\ 2 & -2/3\end{pmatrix}Y}$

Check: $\begin{pmatrix}3 & -1\\ 2 & -2/3\end{pmatrix}$

$(3-\lambda)(-\frac{2}{3}-\lambda)+2=0$

$\cancel{-2}-3\lambda+\frac{2}{3}\lambda+\lambda^2\cancel{+2}=0$

$\lambda^2-3\lambda+\frac{2}{3}\lambda=0$

$\lambda(\lambda-3+\frac{2}{3})=0$

$\lambda=0$ and $\lambda=\frac{7}{3}$

$-\frac{9}{3}+\frac{2}{3}=\frac{-7}{3}$

✓ Excellent!

9. Consider the linear system $\frac{d\mathbf{Y}}{dx} = \begin{pmatrix} -3 & 10 \\ -1 & 3 \end{pmatrix}\mathbf{Y}$. Find a general solution to this system.

eigenvalues

$(-3-\lambda)(3-\lambda) - (-1)(10) = 0$

$-9 + 3\lambda - 3\lambda + \lambda^2 + 10 = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

eigenvectors for $\lambda = i$:

$-3x + 10y = ix$

$-1x + 3y = iy$

$10y = (3+i)x$

$\begin{pmatrix} 10 \\ 3+i \end{pmatrix}$

So $\hat{Y} = e^{it}\begin{pmatrix} 10 \\ 3+i \end{pmatrix}$ is a solution

or $\hat{Y} = (\cos t + i \sin t)\begin{pmatrix} 10 \\ 3+i \end{pmatrix}$

$= \begin{pmatrix} 10\cos t + i\, 10 \sin t \\ 3\cos t - \sin t + i \cos t + 3i \sin t \end{pmatrix}$

$= \begin{pmatrix} 10\cos t \\ 3\cos t - \sin t \end{pmatrix} + i \begin{pmatrix} 10 \sin t \\ \cos t + 3\sin t \end{pmatrix}$

Then by T.C.M.T.

$\hat{Y} = k_1 \begin{pmatrix} 10\cos t \\ 3\cos t - \sin t \end{pmatrix} + k_2 \begin{pmatrix} 10 \sin t \\ \cos t + 3\sin t \end{pmatrix}$

is a general solution

10. Consider the family of linear systems $\frac{d\mathbf{Y}}{dx} = \begin{pmatrix} 3 & b \\ -2 & 1 \end{pmatrix}\mathbf{Y}$.

a) For which values of b will solutions oscillate?

b) For which values of b will solutions move away from the origin for all initial conditions other than (0,0)?

Find eigenvalues:

$$(3-\lambda)(1-\lambda) - (-2)(b) = 0$$

$$3 - 4\lambda + \lambda^2 + 2b = 0$$

$$\lambda^2 - 4\lambda + 2b + 3 = 0$$

So by quadratic:

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2b+3)}}{2(1)} = \frac{4 \pm \sqrt{16 - 8b - 12}}{2}$$

$$= \frac{4 \pm \sqrt{4 - 8b}}{2}$$

$$= 2 \pm \sqrt{1 - 2b}$$

a) Then the eigenvalues are complex when $1 - 2b < 0$, or $b > \frac{1}{2}$, so for those values we have oscillation.

b) For $b \leq \frac{1}{2}$ we have real eigenvalues, which will both be positive (and correspond to solutions moving away from the origin) when

$$2 - \sqrt{1-2b} > 0$$

$$2 > \sqrt{1-2b}$$

$$4 > 1 - 2b$$

$$2b > -3$$

$$b > -3/2$$

Notice that really this includes the $b > \frac{1}{2}$ category as well - sources and spiral sources all have solutions moving outward overall.