Examlet 3 Foundations of Advanced Math 3/23/12

1. a) State the definition of a surjection.

Call a function f surjective iff, $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

Good

b) Give an example of a function from ℕ to ℕ which is injective, ~~and~~ "" make it clear why it is not possible.

$f: \mathbb{N} \to \mathbb{N}$, $f(x) = x$, $x \in \mathbb{N}$.

This is an injective function because $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. $x_1, x_2 \in \mathbb{N}$.

Great

2. a) Let *f* and *g* be bounded functions, both with domain *D*. Then $f + g$ is a bounded function.

$f: D \to \mathbb{R}$ is bounded, so $\exists M$ such that $\forall x \in D$ $|f(x)| \leq M$

$g: D \to \mathbb{R}$ is bounded, so $\exists N$ such that $\forall x \in D$ $|g(x)| \leq N$

$|f(x)| + |g(x)| \leq M + N$ by the Comparison addition Principle.

By the Triangle Equality, we know that

$|f(x) + g(x)| \leq |f(x)| + |g(x)|$

Then by the transitive property of inequalities, we know

$|f(x) + g(x)| \leq M+N.$

Because $f+g$ is still always less than or equal to $M+N$, we know $f+g$ is a bounded function Nice!

a) Let *f* and *g* be bounded functions, both with domain *D*. Then $f - g$ is a bounded function.

$f: D \to \mathbb{R}$ is bounded, so $\exists M$ such that $\forall x \in D$ $|f(x)| \leq M$

$g: D \to \mathbb{R}$ is also bounded, so $\exists N$ such that $\forall x \in D$ $|g(x)| \leq N$.

$|f(x)| + |-g(x)| \leq M+N$ by Comparison Addition Principle

$|f(x) + (-g(x))| \leq |f(x)| + |-g(x)|$ by the Triangle Inequality

By the transitive property of inequalities,

$|f(x) + (-g(x))| \leq M+N$ which is the same as

$|f(x) - g(x)| \leq M+N$

$\therefore$ $f-g$ is a bounded function when both f and g are bounded functions.

Great.

3. If $f{:}A \to B$ and $g{:}B \to C$ are surjective functions, then $g \circ f$ is surjective.

For $g \circ f$ to be surjective, it must be true that $\forall c \in C, \exists a \in A$ such that $g \circ f(a) = c$

Since we know g is surjective, we know $\forall c \in C, \exists b \in B$ such that $g(b) = c$

And since f is surjective, $\forall b \in B, \exists a \in A$ such that $f(a) = b$

For any chosen c, there is a b such that $g(b) = c$

For that particular b, there is an a such that $f(a) = b$

$\therefore g(f(a)) = c$

$g \circ f(a) = c \quad \forall c \in C$

$\therefore g \circ f$ is surjective

Excellent!

4. a) If $f: A \to B$ has an inverse function g, then g has f as an inverse function also.

Well, since $g: B \to A$ is an inverse of f, we know by definition that $\forall a \in A$, $g \circ f(a) = a$ and $\forall b \in B$, $f \circ g(b) = b$.

Now, in order for f to be an inverse of g, $\forall b \in B$ $f \circ g(b) = b$ and $\forall a \in A$, $g \circ f(a) = a$.

Exactly. Since we already know this is true, we know g has f as an inverse when g is an inverse function of f.

b) Let $f: \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 5$ for all $n \in \mathbb{N}$. Find the inverse function of f, or explain why one doesn't exist.

By previous proofs, we know f is invertible iff it is bijective. However, $f(n) = 5$ is not bijective (specifically it is not injective) because $f(1) = f(2) = 5$ but $1 \neq 2$. Thus, f does not have an inverse function. □

Excellent!

5. If A is equipollent to B, and B is equipollent to C, then A is equipollent to C.

Since A is equipollent to B, we know there exists a bijection $f: A \to B$. Since B is equipollent to C, we know there exists a bijection $g: B \to C$. By previous proof, we know when f and g are bijective, $g \circ f: A \to C$ is bijective. Thus, since a bijection exists from A to C, A is equipollent to C. $\square$

Well done!