4 Relations & Graphs

4.1 Relations

Definition: Let A and B be sets. A **relation from A to B** is a subset of $A \times B$. When we have a relation from A to A we often call it a **relation on A**. When we have a relation R on $A \times B$, instead of writing $(x, y) \in R$ we often write $x\ R\ y$.

Example 1: We define a relation N on $\mathbb{R}$ as follows: $N = \{(x, y) \mid x = -y\}$. Then for instance $(-3,3) \in N$, which could also be written $-3\ N\ 3$. This relation is a more formal version of what might be expressed as "is the negative of" as in, "–3 is the negative of 3."

Example 2: We define a relation ~ on $\mathbb{Z}$ as follows: $\sim\, = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid \exists k \in \mathbb{Z}, n - m = 2k\}$. Then for instance $3 \sim 5$ and $6 \sim 20$, but it is not the case that $3 \sim 4$; which might naturally be denoted $3 \nsim 4$. You should recognize that $m \sim n$ when either both values are even or both values are odd.

Exercises 4.1

For each of the relations specified below:
a) Pick an element t of the set in question and find (if possible) three other elements of the set which are related to it.
b) For your element t from part a, find three other elements of the set which are not related to it.

1. Let $\sim$ be the relation on $\mathbb{N}$ defined by $x \sim y$ iff $x - y$ is throdd.
2. Let $\approx$ be the relation on $\mathbb{R}$ defined by $x \approx y$ iff $x - y \in \mathbb{Z}$.
3. Let $\simeq$ be the relation on $\mathbb{R}$ defined by $x \simeq y$ iff $x - y \in \mathbb{Q}$.
4. Let $\cong$ be the relation on $\mathbb{Z}$ defined by $x \cong y$ iff $|x - y| = 5$.
5. Let $\eqcirc$ be the relation on the set $\mathscr{P}(\mathbb{R})$ of all polynomials with real coefficients defined by $f \eqcirc g$ iff f and g have a root in common.
6. Let $\asymp$ be the relation on the set $R(I)$ of integrable functions from [0,1] to $\mathbb{R}$ defined by $f \asymp g$ iff $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$.
7. Let $\succ$ be the relation on the set $\mathbb{R}$ defined by $x \succ y$ iff $x > y + 5$.
8. Let $\ggg$ be the relation on the set $\mathbb{R}$ defined by $x \ggg y$ iff $x > 3y$.
9. Let $\subseteq$ be the usual subset relation on $\mathcal{P}(\mathbb{N})$, the power set of the natural numbers.
10. Let $\bowtie$ be the relation on $\mathcal{P}(\mathbb{N})$ defined by $A \bowtie B$ iff $A \cap B \neq \varnothing$.

4.2 Properties of Relations

Simply put, some relations are more interesting than others. This section explores properties that are often shared by some of the more interesting (and useful) relations used in mathematics.

Definition: A relation ~ on a set S is **reflexive** iff $\forall a \in S$, $a \sim a$.

Definition: A relation ~ on a set S is **symmetric** iff $\forall a, b \in S$, $a \sim b \Rightarrow b \sim a$.

Definition: A relation ~ on a set S is **transitive** iff $\forall a, b, c \in S$, $a \sim b \wedge b \sim c \Rightarrow a \sim c$.

Exercises 4.2

1. Determine whether each of the relations from Exercises 1- 10 in §4.1 is reflexive, symmetric, or transitive.

2. Consider the relation B on the set of all people defined by $a\ B\ b$ iff a is a brother of b. Is this relation reflexive, symmetric, or transitive?

3. Consider the relation S on the set of all people defined by $a\ S\ b$ iff a is a sibling of b. Is this relation reflexive, symmetric, or transitive?

For 4-11, let $S = \{a, b, c\}$. Give an example, if possible, of a relation on S which is:

4. Reflexive, symmetric, transitive.

5. Reflexive, symmetric, not transitive.

6. Reflexive, not symmetric, transitive.

7. Reflexive, not symmetric, not transitive.

8. Not reflexive, symmetric, transitive.

9. Not reflexive, symmetric, not transitive.

10. Not reflexive, not symmetric, transitive.

11. Not reflexive, not symmetric, not transitive.

12. If two relations R and S on A are reflexive, is $R \cup S$ reflexive? $R \cap S$?

13. If two relations R and S on A are symmetric, is $R \cup S$ symmetric? $R \cap S$?

14. If two relations R and S on A are transitive, is $R \cup S$ transitive? $R \cap S$?

4.3 Equivalence Relations

As noted in the last section, some relations are more useful than others. Particularly useful are ones with some common characteristics that in turn lead to other properties, as developed below.

Definition: A relation which is reflexive, symmetric, and transitive is called an **equivalence relation**.

Example 1: Congruence of triangles, which should be familiar to you from high school geometry, is an equivalence relation.

Example 2: We defined congruence modulo n in section 1.3; note that Exercises 14-16 in that section amount to a proof that this gives an equivalence relation on $\mathbb{Z}$.

Example 3: Exercises 1-3 in section 3.6 prove that equipollence provides an equivalence relation on the set of all subsets of any universal set.

Definition: Given a set S, an element $a \in S$, and an equivalence relation $\sim$ on S, the set of elements of S that are related to a is called the **equivalence class of *a*** and denoted $[a]$, i.e. $\mathbf{[a]} = \{x \in S \mid x \sim a\}$.

Example 4: If we consider, as in Example 2 above, the equivalence relation of congruence modulo 2, then the equivalence class of 1 includes 3 since $3 - 1 = 2$, and $2 = 2 \cdot 1$ where $1 \in \mathbb{Z}$. In fact, $[1]$ is exactly the odd integers.

Example 5: For the equivalence relation of equipollence on some collection of sets mentioned in Example 3 above, the equivalence class of a set with, say, 2 elements will be all of the other sets having exactly 2 elements. If the collection of sets included $\mathbb{R}$ and all of its subsets, then the equivalence class of $\mathbb{R}$ itself would include the set of irrationals (among many others), and the equivalence class of $\mathbb{N}$ would include $\mathbb{Z}$, and $\mathbb{Q}$ (among many others).

Definition: A set C of sets is **pairwise disjoint** iff $\forall S_1, S_2 \in C$, $S_1 \cap S_2 = \varnothing$ or $S_1 = S_2$.
Definition: A **partition** of a set S is a set of non-empty, pairwise disjoint subsets of S whose union is all of S.

Theorem 1: Let S be a set and $\wp$ a partition of S. The relation on S defined by $a \sim b$ iff $\exists P \in \wp$ for which $a, b \in P$ is an equivalence relation.

Proof: Exercises 10-12.

Theorem 2: Let S be a set and $\sim$ be an equivalence relation on S. The set $\{[a] \mid a \in S\}$ is a partition of S.

Proof: Exercise 13.

Exercises 4.3

1. Consider the relation $|||$ on $\mathbb{Z}$ defined by $a \,|||\, b$ iff $a - b$ is threven. Determine whether $|||$ is an equivalence relation.

2. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ iff $|a| = |b|$. Determine whether $\sim$ is an

equivalence relation.

3. Consider the relation ≈ on ℝ × ℝ defined by $(x_1, y_1) \approx (x_2, y_2)$ iff $x_1^2 + y_1^2 = x_2^2 + y_2^2$. Determine whether ≈ is an equivalence relation.

4. Consider the relation ↗ on ℝ × ℝ defined by $(x_1, y_1) \nearrow (x_2, y_2)$ iff $x_1 > x_2 \vee (x_1 = x_2 \wedge y_1 \geq y_2)$. Determine whether ↗ is an equivalence relation.

5. For any of the relations in Exercises 1-4 which are equivalence relations, describe the equivalence classes.

6. A collection C of sets S_i indexed by some set I is pairwise disjoint iff $\bigcap_{i \in I} S_i = \varnothing$.

7. Let $S = \{1, 2, 3, 4\}$. Then $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$ is a relation on S. Write the equivalence classes of S associated with R.

8. Let $S = \{1, 2, 3, 4\}$. Suppose that ~ is an equivalence relation with $1 \sim 2 \sim 3$, but $1 \nsim 4$. What is the partition associated with ~?

9. Let $S = \{1, 2, 3, 4\}$. Suppose that ~ is an equivalence relation with $1 \sim 4$. What are the possible partitions associated with ~?

10. Show that the relation defined in Theorem 1 is reflexive.

11. Show that the relation defined in Theorem 1 is symmetric.

12. Show that the relation defined in Theorem 1 is transitive.

13. Prove Theorem 2.

4.4 Functions as Relations

Chapter 3 was devoted to functions (and their implications via equipollence), but only gave a limited version of what functions actually *are*. This section gives the mature version.

> Definition: A relation R from A to B is a **function from A to B** iff for all $a \in A$ there exists a unique $b \in B$ for which $(a, b) \in R$. Due to this uniqueness we often write $\boldsymbol{R(a) = b}$ when $(a, b) \in R$.

While technically it's improper to give a new definition of a term that was previously defined (in section 3.1), in fact our previous definition was vague on exactly the point this new version allows us to make precise. Before there was reference to a "rule" without any specification of what might be allowable as a rule (can you roll dice, for instance?). This new version specifies, and in terms of material (set theory) that we have previously developed.

Example 1: Consider $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = y\}$ as a function from $\mathbb{R}$ to $\mathbb{R}$. This is the standard parabola familiar to all Calc 1 students, but presented in a more formal manner.

Example 2: Consider $i_A = \{(a, a) \mid a \in A\}$ as a function from A to A. This is the identity function on A defined in section 3.1, now presented in a more sophisticated way.

Exercises 4.4

1. Express the constant function $f(x) = c$ from $\mathbb{R}$ to $\mathbb{R}$ formally as a set of ordered pairs.
2. Express the inclusion function from A to B formally as a set of ordered pairs.
3. Express the definition of an even function in terms of ordered pairs.
4. Express the definition of an odd function in terms of ordered pairs.
5. Express the definition of an increasing function in terms of ordered pairs.
6. Express the definition of a decreasing function in terms of ordered pairs.
7. Express the definition of a bounded function in terms of ordered pairs.
8. Express the definition of the sum of two functions in terms of ordered pairs.
9. Express the definition of the product of two functions in terms of ordered pairs.
10. Express the definition of the composition of two functions in terms of ordered pairs.
11. Express the definition of a surjective function in terms of ordered pairs.
12. Express the definition of an injective function in terms of ordered pairs.
13. Express the definition of an inverse function in terms of ordered pairs.
14. Suppose that f and g are functions from A to B. Is $f \cup g$ a function from A to B? $f \cap g$?

4.5 Graph Theory

First, a disclaimer: the graphs in the section are not the graphs of Calculus or high school algebra, but instead the graphs involved in the area of mathematics and computer science called graph theory. Although we will not explore this area in much depth, it fits naturally with the rest of the material in this chapter and is so widely seen in applications of mathematics to interesting situations that any serious math student should be acquainted with it.

Definition: A **graph** G is a set V of vertices along with a set E of edges, where each edge is a set containing exactly two vertices.

You should note that this is almost identical to the definition of a relation on a set V – see Exercise 3.

Definition: The **degree** of a vertex v in a graph, sometimes denoted $d(v)$, is the number of edges containing v.

Definition: A **walk** (or a **walk of length *n***) is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$, alternating vertices and edges, starting and ending with vertices, and where each edge is adjacent to the preceding and succeeding vertices. Note that it is standard to refer to v_0 as a walk (of length 0), since it can be thought of as vacuously satisfying the alternation of vertices and edges. ~~A **path** is a walk in which all the vertices are distinct.~~ A **cycle** is a walk in which $v_0 = v_n$ and $n \geq 3$. A **vertex-disjoint walk** is a walk in which no vertices are repeated. A **vertex-disjoint cycle** is a cycle in which $v_0 = v_n$ and no other vertices are repeated. An **edge-disjoint walk** is a walk in which no edges are repeated, and an **edge-disjoint cycle** is a cycle in which no edge is repeated.

Definition: A graph G is said to be **connected** iff every pair of vertices is joined by a walk.

Definition: A **tree** is a connected graph with no vertex-disjoint cycles.

Exercises 4.5

1. How many different graphs are there with three vertices?

2. How many different graphs are there with four vertices?

3. Describe the connection between a relation R on a set V and a graph G with the set V of vertices.

4. Create a relation based on a graph G by saying elements $v_1, v_2 \in V$ are related iff there exists a walk from v_1 to v_2. Is the resulting relation reflexive? Symmetric? Transitive?

5. The maximum possible number of edges in a graph with v vertices is __________.

6. The number of edges in a tree with v vertices is __________.

7. The minimum number of edges in a connected graph with v vertices is __________.

8. The sum of the degrees of the vertices in a graph is twice the number of edges in the graph.

9. In any graph, the number of vertices of odd degree is even.

10. A graph with all vertices of degree 3 is **cubic**. Every cubic graph has an even number of vertices.

11. The maximum number of vertices of degree 1 in a tree with v vertices is __________.

12. The minimum number of vertices of degree 1 in a tree with v vertices is __________.

13. The maximum number of vertices of degree 2 in a tree with v vertices is __________.

14. The minimum number of vertices of degree 2 in a tree with v vertices is __________.

15. Find all trees with $v \leq 6$.

16. Find all trees with $v = 7$.

17. Find all trees with $v = 8$.

18. Any graph where each vertex has degree at least 2 is connected.

19. Any graph where each vertex has degree at least 3 is connected.

20. Any graph where each vertex is part of a cycle is connected.

21. Any graph where some vertex v_0 is joined to each other vertex in the graph by a walk is connected.

22. If a graph G is connected, then the graph G' having the same vertex set and an edge set with exactly one fewer element is also connected.

23. If a graph G is connected, then the graph G' having a vertex set with exactly one fewer element and an edge set with all elements not including that removed vertex is also connected.

24. An **Euler path** in a graph G is a walk which passes through each edge of G exactly once. An Euler path exists in a connected graph G iff there are exactly two vertices of odd degree in G.

25. A **planar graph** is one which can be drawn in the plane without any edges crossing each other. In any planar, connected graph with v vertices and e edges, where $v \geq 3$, $e \leq 3v - 6$.

26. In any planar, connected graph with v vertices and e edges, dividing the plane into f distinct regions (including the exterior), $v - e + f = 2$.

27. Let $\boldsymbol{K_n}$ denote the **complete graph on *n* vertices**, a graph with n vertices and edges connecting every pair of vertices. Find the number of edges in K_n.

28. K_4 is planar.

29. K_5 is planar.

30. A **subgraph** H of a graph G is a graph with the vertex set of H a subset of the vertex set of G, and the edge set of H all edges from G for which both vertices are in the vertex set of H. A

subgraph of a connected graph is connected.

31. If a graph *G* has all of its subgraphs connected, then *G* is connected.

32. A subgraph of a planar graph is planar.

33. If a graph *G* has all of its subgraphs planar, then *G* is planar.

34. A **spanning tree** for a graph *G* is a subgraph of *G* which is a tree containing all vertices of *G*. Any graph *G* has a spanning tree.

35. Any connected graph has a unique spanning tree.

36. Any graph with a spanning tree is connected.