5 Counting

5.1 Combinatorics

The area of mathematics dealing with counting (sometimes very large) collections is called Combinatorics. This section gives a very brief introduction to the basic tools of Combinatorics, and the next section puts some of these to use.

Theorem (The Multiplication Principle): If one choice can be made in m different possible ways and a second choice can be made in n different possible ways, then the two choices can be made in $m \cdot n$ total ways.

Proof: Exercise 1.

Definition: Let $\mathbb{Z}_n = \{m \in \mathbb{Z} \mid 0 \le m < n\}$.

Definition: Let A be a set with n elements. A bijection $f : \mathbb{Z}_n \to A$ is called a **permutation**.

Definition: Let A be a set with n elements. A **permutation of size *k* of *n* elements** is an injection from $\mathbb{Z}_k$ to A. We denote the number of these distinct injections $\boldsymbol{P(n, k)}$.

Definition: The number of distinct k-element subsets of a set with n elements is denoted $\boldsymbol{C(n, k)}$ or $\binom{n}{k}$, read "n choose k".

Exercises 5.1

1. Prove The Multiplication Principle.
2. The number of distinct permutations of a set A having n elements is $n!$
3. Find the number of orders in which 6 books can be arranged on a shelf.
4. Find the number of orders in which a deck of 52 cards can be arranged.
5. $P(n, k) = \dfrac{n!}{(n-k)!}$.
6. Find the number of orders in which 3 runners can finish first, second, and third out of 10.
7. Find the number of orders in which 3 letters from the set {a, e, i, o, u} can be chosen.
8. $P(n, n) = n!$
9. For $n \ge 1$, $P(n, n) = P(n, n-1)$.
10. For $n \ge 1$ and $1 \le k \le n$, $C(n, k) = \dfrac{n!}{k!(n-k)!}$.

11. Find the number of 5-card poker hands that can be dealt from a deck of 52.

12. Find the number of ways 2 faculty members can be chosen from a department of 5 to represent the department at a reception for prospective students.

13. For $n \geq 0$, $C(n, 0) = 1 = C(n, n) = 1$.

14. For $n \geq 1$, $C(n, 1) = n$.

15. For $n \geq 1$ and $1 \leq k \leq n$, $C(n + 1, k) = C(n, k) + C(n, k - 1)$.

16. For $n \geq 1$ and $1 \leq k \leq n$, $k \cdot C(n, k) = n\, C(n - 1, k - 1)$.

17. For $n \geq 1$ and $1 \leq k \leq n$, $(n - k) \cdot C(n, k) = n \cdot C(n - 1, k)$

18. For $n \geq 1$ and $0 \leq k \leq n$, $C(n, k) = C(n, n - k)$.

19. The Banana Theorem: Let A be a set with n elements of k different types (such that elements of the same type are regarded as indistinguishable from one another for purposes of orderings). Let n_i be the number of elements of type i for each integer i from 1 to k. Then the number of different arrangements of the elements in A will be $\dfrac{n!}{\prod_{i=1}^{k}(n_i!)}$.

20. How many distinguishable ways can the letters in the word banana be arranged?

21. How many distinguishable ways can the letters in the word Mississippi be arranged?

22. How many numbers between 1 and 1,000,000 have no 9's among their digits?

5.2 Probability

This section introduces the serious study of probability, both because every math student should have some familiarity with it and because it provides perfect uses for the counting tools introduced in the previous section.

Definition: $\textbf{Probability} = \dfrac{\text{number of favorable events}}{\text{number of total events}}$.

Example 1: If a standard die is tossed, the probability of rolling a 2 or less is 1/3, because the number of favorable events is 2 (the number of elements in the set {1,2}) and the number of total possible events is 6 (the number of elements in the set {1,2,3,4,5,6}), and we can reduce 2/6 to 1/3.

Example 2: If two standard dice are tossed, the probability that the second one is at least twice the first is 1/4, because the number of favorable events is 9 (indicated by the boxed entries in the table below) and the number of total possible events is 36 (indicated by all of the entries in the table below), and 9/36 reduces to 1/4.

$$\begin{array}{cccccc}
1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\
\boxed{2,1} & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\
\boxed{3,1} & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\
\boxed{4,1} & \boxed{4,2} & 4,3 & 4,4 & 4,5 & 4,6 \\
\boxed{5,1} & \boxed{5,2} & 5,3 & 5,4 & 5,5 & 5,6 \\
\boxed{6,1} & \boxed{6,2} & \boxed{6,3} & 6,4 & 6,5 & 6,6
\end{array}$$

Exercises 5.2

1. When a fair coin is tossed, what is the probability of the coin coming up tails?

2. When a fair coin is tossed twice, what is the probability of two heads?

3. When a fair coin is tossed twice, what is the probability of one head and one tail?

4. When a fair coin is tossed three times, what is the probability of exactly two tails?

5. When a fair coin is tossed 100 times, what is the probability of exactly two tails?

6. A coin is tossed 3 times. What is the probability of all three tosses being heads if you know that at least one of the tosses came up heads?

7. If a jar contains four balls, one red, two blue, and one white, and two balls are drawn at random from the jar (without replacement), what is the probability that neither ball is blue?

8. If a jar contains four balls, one red, two blue, and one white, and two balls are drawn at random from the jar (without replacement), what is the probability that neither ball is blue if you know that the white ball is one of the two drawn?

9. If a jar contains five balls, one red, two blue, and two white, and three balls are drawn at random from the jar (replacing each after it's drawn), what is the probability that the balls are drawn red, blue, and white in that order?

10. If a jar contains five balls, one red, two blue, and two white, and three balls are drawn at random from the jar (replacing each after it's drawn), what is the probability that the balls are drawn red, blue, and white, not necessarily in that order?

11. If two cards are drawn from a standard deck, what is the probability of both cards being clubs?

12. If two cards are drawn from a standard deck, what is the probability of both cards being face cards?

13. If two cards are drawn from a standard deck, what is the probability of both cards being red aces?

14. If a standard die is tossed, what is the probability that the result is at least 5?

15. If a standard die is tossed twice, what is the probability that the total is exactly 5?

16. What is the probability that when a die is tossed twice, the difference between the two rolls is exactly two?

17. A standard die is tossed twice. What is the probability that the second roll is higher than the first?

18. If a standard die is tossed three times consecutively, what is the probability that each result is higher than the one before?

19. If three standard dice are tossed, what is the probability that the total on all three dice is at least 17?

20. If someone knows your password is a six-character string of letters, what is their probability of guessing it in a single try?

21. If someone knows your password is a case-sensitive six-character string of letters, what is their probability of guessing it in a single try?

22. If someone knows your password is a case-sensitive six-character string of letters and numerals, what is their probability of guessing it in a single try?

23. If someone knows your password is a six-character string of letters, and that it has a "g" in it, what is their probability of guessing it in a single try?

24. If someone knows your password is a case-sensitive six-character string of letters and numerals, and that at least one of the letters was required to be capitalized, what is their probability of guessing it in a single try?

5.3 The Peano Axioms I

You should have wondered at some point in your life exactly how many things needed to be accepted "on faith" as just obviously true in order to make the rest of the things commonly done in mathematics follow. The answer, technically, is none – mathematics proceeds in a hypothetical way, deducing the consequences *if* some collection of axioms holds. But for a more satisfying answer, this and the following sections provide an instance of developing a great deal of familiar material from a very short list of principles.

Definition: We call a set N a **Peano system** iff the following conditions are satisfied:

P1. $0 \in N$.
P2. For each $x \in N$, there is a unique element $x' \in N$ (we call x' the **successor** of x).
P3. $\forall x \in N, x' \neq 0$.
P4. $\forall x, y \in N, x' = y' \Rightarrow x = y$
P5. If $M \subseteq N$ for which $0 \in M$ and $\forall x \in M, x' \in M$, then $M = N$.

Definition: Given a Peano system N and $x, y \in N$, define their **sum** $x + y$ by

A1. $x + 0 = x$
A2. $x + y' = (x + y)'$

Exercises 5.3

Prove the following statements, given that N is a Peano system.

1. $\forall x, y \in N, x + (y + 0) = (x + y) + 0$.

2. $\forall x, y, z \in N, x + (y + z) = (x + y) + z \Rightarrow x + (y + z') = (x + y) + z'$.

3. $\forall x, y, z \in N, x + (y + z) = (x + y) + z$.

4. $0 + 0 = 0$.

5. $\forall y \in N, 0 + y = y \Rightarrow 0 + y' = y'$.

6. $\forall y \in N, 0 + y = y$.

7. $\forall x \in N, x' + 0 = (x + 0)'$.

8. $\forall x, y \in N, x' + y = (x + y)' \Rightarrow x' + y' = (x + y')'$.

9. $\forall x, y \in N, x' + y = (x + y)'$.

10. $\forall y \in N, 0 + y = y + 0$.

11. $\forall x, y \in N, x + y = y + x \Rightarrow x' + y = y + x'$.

12. $\forall x, y \in N, x + y = y + x$.

5.4 The Peano Axioms II

The previous section developed the basic additive properties of the natural number system. This section extends that development to multiplication.

Definition: Given a Peano system N and $x, y \in N$, define their **product** $x \cdot y$ by

M1. $x \cdot 0 = 0$

M2. $x \cdot y' = (x \cdot y) + x$

Exercises 5.4

Prove the following statements, given that N is a Peano system.

1. $\forall x, y \in N, x \cdot (y + 0) = x \cdot y + x \cdot 0$.
2. $\forall x, y, z \in N, x \cdot (y + z) = x \cdot y + x \cdot z \Rightarrow x \cdot (y + z') = x \cdot y + x \cdot z'$.
3. $\forall x, y, z \in N, x \cdot (y + z) = x \cdot y + x \cdot z$.
4. $\forall x, y \in N, x \cdot (y \cdot 0) = (x \cdot y) \cdot 0$.
5. $\forall x, y, z \in N, x \cdot (y \cdot z) = (x \cdot y) \cdot z \Rightarrow x \cdot (y \cdot z') = (x \cdot y) \cdot z'$.
6. $\forall x, y, z \in N, x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
7. $0 \cdot 0 = 0$.
8. $\forall y \in N, 0 \cdot y = 0 \Rightarrow 0 \cdot y' = 0$.
9. $\forall y \in N, 0 \cdot y = 0$.
10. $\forall x \in N, x' \cdot 0 = 0$.
11. $\forall x, y \in N, x' \cdot y = x \cdot y + y \Rightarrow x' \cdot y' = x \cdot y' + y'$.
12. $\forall x, y \in N, x' \cdot y = x \cdot y + y$.
13. $\forall y \in N, 0 \cdot y = y \cdot 0$.
14. $\forall x, y \in N, x \cdot y = y \cdot x \Rightarrow x' \cdot y = y \cdot x'$.
15. $\forall x, y \in N, x \cdot y = y \cdot x$.

5.5 The Peano Axioms III

Here we include a few additional results that can be developed from the axioms already given. While it is impossible to list all the consequences of these axioms, these should dispel any impression that the results given previously are exhaustive.

Definition: We henceforth adopt the convention that $0' = 1$, $1' = 2$, $2' = 3$, $3' = 4$, and so forth.

Exercises 5.5

Prove the following statements, given that N is a Peano system.

1. $\forall y \in N$, with $y \neq 0$, $0 \neq 0 + y$.

2. $\forall x, y \in N$, with $y \neq 0$, $x \neq x + y \Rightarrow x' \neq x' + y$.

3. $\forall x, y \in N$, with $y \neq 0$, $x \neq x + y$.

4. $\forall y, z \in N$, $0 + y = 0 + z \Rightarrow y = z$.

5. $\forall x, y, z \in N$, $(x + y = x + z \Rightarrow y = z) \Rightarrow (x' + y = x' + z \Rightarrow y = z)$.

6. $\forall x, y, z \in N$, $x + y = x + z \Rightarrow y = z$.

7. $\forall y \in N$, with $y \neq 1$, $1 \neq 1 \cdot y$.

8. $\forall x, y \in N$, with $y \neq 1$, $x \neq x \cdot y \Rightarrow x' \neq x' \cdot y$.

9. $\forall x, y \in N$, with $y \neq 1$, $x \neq x \cdot y$.

10. $\forall y, z \in N$, $1 \cdot y = 1 \cdot z \Rightarrow y = z$.

11. $\forall x, y, z \in N$, $(x \cdot y = x \cdot z \Rightarrow y = z) \Rightarrow (x' \cdot y = x' \cdot z \Rightarrow y = z)$.

12. $\forall x, y, z \in N$, $x \cdot y = x \cdot z \Rightarrow y = z$.

5.6 The Peano Axioms IV

A sceptic might look at the previous sections and worry that, although they possess internal consistency, there is no particular reason to accept the axioms in their somewhat convoluted form as self-evident. Said differently, whether nine things or a thousand are taken on faith, we still might like to think mathematics is based on something other than hypotheticals. This section addresses that concern.

Definition: Given a set A, define $\boldsymbol{S(A)} = A \cup \{A\}$.

Definition: Let $\mathfrak{N}$ be the collection of all sets N containing $\varnothing$ and such that $\forall x \in N$, $S(x) \in N$. Let $\mathbb{N}$ be the intersection of all sets in $\mathfrak{N}$.

Exercises 5.6

1. Write $S(\varnothing)$, $S(S(\varnothing))$, and $S(S(S(\varnothing)))$ explicitly. How many elements does each of these have?
2. Show that $\varnothing \in \mathbb{N}$.
3. Show that for each $x \in \mathbb{N}$, there is a unique element $\mathrm{S}(x) \in \mathbb{N}$.
4. Show that for any set A, $S(A) \neq \varnothing$.
5. Show that for any x, $\mathrm{y} \in \mathbb{N}$, $S(x) = S(y) \Rightarrow x = y$
6. Show that if $M \subseteq \mathbb{N}$ for which $\varnothing \in M$ and $\forall x \in M$, $S(x) \in M$, then $M = \mathbb{N}$.
7. Show that $\mathbb{N}$ is a Peano system.
8. Show that $1 + 1 = 2$.
9. Show that $2 + 2 = 4$.