Examlet 1B **Foundations of Advanced Math** **2/7/13**

1. Show that the square of a throddodd integer is throdd.

Let n be throddodd, so n can be written as $3k+2$, $k \in \mathbb{Z}$

$n^2 = (3k+2)^2$

$n^2 = 9k^2 + 12k + 4$

$n^2 = 9k^2 + 12k + 3 + 1$

$n^2 = 3(3k^2 + 4k + 1) + 1$

where $(3k^2 + 4k + 1) \in \mathbb{Z}$ by Closure of Integers

Thus n^2 is throdd by definition of throdd $(3l+1, l \in \mathbb{Z})$. $\square$

Great!

2. Determine whether $(P \Rightarrow Q) \vee (P \Rightarrow R)$ is logically equivalent to $P \Rightarrow (Q \vee R)$.

P	Q	R	$P \to Q$	$P \to R$	$Q \vee R$	$(P \to Q) \vee (P \to R)$	$P \to (Q \vee R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

Since, the truth values for columns $(P \to Q) \vee (P \to R)$ and $P \to (Q \vee R)$ are identical for all possibilities then they are logically equivalent. $\square$

Excellent

3. a) If $p \equiv_6 1$, then $p \equiv_3 1$.

Proof:

$6 \mid (1-p)$ definition of congruence

for some $a \in \mathbb{Z}$, $1-p = a \cdot 6$. definition of divisibility

$1-p = 2a \cdot 3$

$2a \in \mathbb{Z}$ by closure of integers

$\therefore 3 \mid (1-p)$

$\therefore p \equiv_3 1$. $\square$

Nice Job

b) If $p \equiv_3 1$, then $p \equiv_6 1$.

This is not true.

Counterexample: $16 \equiv_3 1$ but $16 \equiv_6 4$ and $16 \not\equiv_6 1$. $\square$

Great!

4. $\sqrt{3}$ is irrational.

Suppose $\sqrt{3}$ is rational.

$\sqrt{3} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and are relatively prime (no common factors) and $q \neq 0$.

$3 = \frac{p^2}{q^2}$

$3q^2 = p^2$

$p^2, q^2 \in \mathbb{Z}$ by closure

p^2 is threven by definition of threven.

We know that throdd and throddodd numbers squared are throdd, so p must be threven as well.

$p = 3r$, $r \in \mathbb{Z}$.

$3q^2 = (3r)^2 = 9r^2$

0 $q^2 = 3r^2$

q^2 is threven by definition.

Therefore, q is threven as well.

p and q have a common factor, 3

Great! We have reached a contradiction.

$\therefore \sqrt{3}$ is irrational. $\square$

5. For all $n \in \mathbb{N}$, $n^2 \geq n$.

Let's induct!

First our base case, when $n=0$, is

$$(0)^2 \geq (0), \quad \text{or} \quad 0 \geq 0,$$

which is true.

Now s'pose it's true for some $n=k$, so $k^2 \geq k$.

We want to show $(k+1)^2 \geq (k+1)$.

We know $k^2 \geq k$, and also that $k \geq 0$, so $2k \geq 0$.

Adding these gives $k^2+2k \geq k$, and adding 1 to each side gives $k^2+2k+1 \geq k+1$, and factoring this is $(k+1)^2 \geq k+1$, as desired.

So since it was true for our base case, and if it's true for $n=k$ it will also be true for $n=k+1$, then by mathematical induction it's true for all $n \geq 0$. $\square$