Examlet 3a   Foundations of Advanced Math   3/29/13

1. a) State the definition of a surjection.

$f: A \to B$ is surjective iff $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

Good

b) Give an example of a function from ℕ to ℕ which is injective, or make it clear why it is not possible.

$f(x) = x$.

(It is injective because every natural ~~#~~ number yields its own individual natural number. It also happens to be surjective but $f(x)$ could also equal $2x$ which is not surjective.)

Excellent.

2. a) Let f and g be bounded functions, both with domain D. Then $f + g$ is a bounded function.

Since $f(x)$ is bounded, this means for all values of x,
$|f(x)| \leq M$ where $M \in \mathbb{R}$.
Since $g(x)$ is bounded, this means for all values of x,
$|g(x)| \leq A$ where $A \in \mathbb{R}$.
By addition of inequalities, $|f(x)| + |g(x)| \leq M + A$.
We know from the Triangle Inequality that $|f(x) + g(x)| \leq |f(x)| + |g(x)|$
So, by the transitive property $|f(x) + g(x)| \leq M + A$
Since this is true for all values of x, and $(M+A) \in \mathbb{R}$,
$f+g$ is a bounded function by definition $\square$

Good

a) Let f and g be bounded functions, both with domain D. Then $f - g$ is a bounded function.

Since $f(x)$ is bounded, for all values of x $|f(x)| \leq M$
where $M \in \mathbb{R}$.
Since $g(x)$ is bounded, for all values of x, $|g(x)| \leq A$
where $A \in \mathbb{R}$. Since this is an absolute value function,
$|g(x)| = |-g(x)|$ so $|-g(x)| \leq A$.
By addition of inequalities, $|f(x)| + |-g(x)| \leq M + A$.
By the triangle inequality, $|f(x) + -g(x)| \leq |f(x)| + |-g(x)|$
So by the transitive property, $|f(x) - g(x)| \leq M + A$
Since this is true for all values of x and $(M+A) \in \mathbb{R}$,
$f-g$ is a bounded function by definition $\square$

Great.

3. If $f: A \to B$ and $g: B \to C$ are surjective functions, then $g \circ f$ is surjective.

Since g is a surjective function, $\forall c \in C$, $\exists b \in B$ such that $g(b) = c$.

Since f is surjective, $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

By composition, $g(f(a)) = g(b) = c$ by substitution.

So, $g(f(a)) = c$

For all values of $c \in C$, there will exist an element in B, for this element b (and all elements of B), there exists an element a in A. such that $g(f(a)) = c$.

Since $\forall c \in C$, $\exists a \in A$ such that $(g \circ f)(a) = c$, $g \circ f$ is surjective by definition $\square$

Good

4. a) If $f: A \to B$ has an inverse function g, then g has f as an inverse function also.

$f: A \to B$ has an inverse function g.

So, by definition of inverse $(f \circ g)(b) = b$ and $(g \circ f)(a) = a$.

So the inverse of $f: A \to B$ is $g: B \to A$. g will be invertible iff there exists a function, which we will call h, such that h has a domain of A and a codomain of B and $(g \circ h)(a) = a$ and $(h \circ g)(b) = b$.
Well, as stated above $f: A \to B$ fulfills these requirements, so $(g \circ f)(a) = a$ and $(f \circ g)(b) = b$

$\therefore$ f is an inverse of g $\square$

Good

b) Let $f: \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 5$ for all $n \in \mathbb{N}$. Find the inverse function of f, or explain why one doesn't exist.

The inverse of the function $f: \mathbb{N} \to \mathbb{N}$ does not exist.

Because $f(n) = 5$ for all $n \in \mathbb{N}$ $f(1) = 5$ and $f(2) = 5$, so $f(1) = f(2)$, but $1 \neq 2$, so it is not true that $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$; therefore f is not injective, and (as previously proven), in order for a function to be invertible, it must be bijective, and, since f is not injective, f is not bijective, and therefore not invertible $\square$

Great.

5. If A is equipollent to B, and B is equipollent to C, then A is equipollent to C.

Since A is equipollent to B, there exists a bijection $f: A \to B$
Since B is equipollent to C, there exists a bijection $g: B \to C$.
By composition, $(g \circ f): A \to C$ is also a bijection.
Since a bijection exists with domain A and codomain C,
then by definition A is equipollent to C $\square$.

Excellent!