**Examlet 3b** **Foundations of Advanced Math** 3/29/13

1. a) Give an example of a function (specifying its domain and codomain) which is even.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ for $x \in \mathbb{R}$ is even

Yes.

b) Let $f$ and $g$ be bounded functions, both with domain $D$. Then $f+g$ is a bounded function.

$f: D \to \mathbb{R}$ is bounded, $\exists M \in \mathbb{R}$ that $\forall x \in D$, $|f(x)| \leq M$

$g: D \to \mathbb{R}$ is bounded, $\exists O \in \mathbb{R}$ that $\forall x \in D$ $|g(x)| \leq O$

from previous work we can show that $|f(x)| + |g(x)| \leq M + O$

from the Triange Inequality we can show $|f(x) + g(x)| \leq |f(x)| + |g(x)|$

using the Transitive property we can find $|f(x) + g(x)| \leq M + O$ or

$|(f+g)(x)| \leq M + O$

Since $(M+O) \in \mathbb{R}$ by closure of the reals

$(f+g)(x)$ is a bounded function $\square$

Excellent!

2. a) State the definition of a surjection.

We say that $f: A \to B$ is a surjective function iff $\underline{\forall b \in B}$, $\underline{\exists a \in A}$ such that $\underline{f(a) = b}$.

Good

b) Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ which is surjective, but not injective.

$$f: \mathbb{N} \to \mathbb{N}, \quad f(x) = \begin{cases} x-1 & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

This function is surjective because all values of the codomain have a corresponding value in the domain. However, it is not injective because $f(0) = 1 = f(2)$ but $0 \neq 2$. $\square$

Excellent!

3. If $f$:$A \to B$ and $g$:$B \to C$ are injective functions, then $g \circ f$ is injective.

Let $g \circ f(x) = h(x)$.

Suppose $h(a_1) = h(a_2)$, this is to say $g \circ f(a_2) = g \circ f(a_1)$.
but since $g$ is injective $g \circ f(a_1) = g \circ f(a_2) \Rightarrow f(a_1) = f(a_2)$
and since $f$ is injective $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.
Since $h(a_1) = h(a_2) \Rightarrow a_1 = a_2$, $h(x)$, or $g \circ f(x)$, must be injective. □

Nice!

4. If $f$: $A \to B$ has an inverse function $g$, then $g$ has $f$ as an inverse function also.

Since $f: A \to B$ has an inverse function $g: B \to A$

$\forall a \in A \; g(f(a)) = a$ and $\forall b \in B \; f(g(b)) = b$, this is logically equivalent to

$\forall b \in B \; f(g(b)) = b$ and $\forall a \in A \; g(f(a)) = a$, which is the definition of $g$ having an inverse function $f$.

Therefore $g$ has an inverse function $f$. □

Excellent!

5. a) Any two countable sets are equipollent.

Let's suppose a set $A = \{1\}$, which is countable. Now let's suppose a set $B = \{2,3\}$, which is also countable. There is no way to make a surjective function $f: A \to B$, and therefore no bijection exists from A to B. Since no bijection exists, A and B are not equipollent meaning that the statement is false by counterexample. □

Excellent

b) Any two denumerable sets are equipollent.

If A is a denumerable set, then there exists a bijection $f: A \to \mathbb{N}$. If B is a denumerable set, then there exists a bijection $g: B \to \mathbb{N}$. By an earlier proof this means that there exists a bijection $h: \mathbb{N} \to B$. Now suppose $h \circ f: A \to B$. This new function is bijective by an earlier proof. Therefore there exists a bijection from A to B meaning that two denumerable sets are equipollent. □

Nice!