Generalized Parabolas

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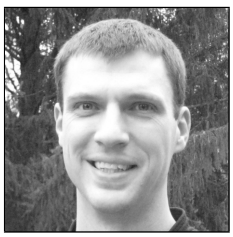

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A parabola is the locus of all points equidistant from a given point (called the *focus*) and a line (called the *directrix*). It is easy to be content with this situation: a line + a point = a parabola. However, an annoying habit of mathematicians is that we constantly tweak things. So, the authors asked: "What happens if the directrix is not a line? What curves, if any, result when we look at all points equidistant from a given point (the focus) and a *curve* (the *generalized directrix*)?" What *generalized* parabolas will result?

A second annoying habit of mathematicians is that we tend to report our findings in a manner that completely obfuscates the wonder of the process. In this paper we will try to glory in the tweaking while avoiding the obfuscation.

Finding generalized parabolas

To answer the questions stated above, we introduce some notation. We denote the focus by $p = (x_0, y_0)$ and define the directrix parametrically, $\vec{r}(t) = \langle f(t), g(t)\rangle$, for some functions f and g. We assume initially that p does not lie on $\vec{r}$. The resulting *generalized parabola*, assuming there is one, will also be defined parametrically, $\vec{q}(t) = \langle x(t), y(t)\rangle$, for some yet to be discovered functions x and y.

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The parametrization of $\vec{r}$ gives us a handy way of finding points equidistant from the directrix and the focus, i.e., finding points on $\vec{q}$. Given a point on $\vec{r}$, say $\vec{r}(t_0)$, draw the line l perpendicular to the directrix at t_0 along which we measure distance to the directrix (here we assume that $\vec{r}$ is smooth at t_0). We seek a point $\vec{q}(t_0)$ on l that is equidistant from p and $\vec{r}(t_0)$. The locus of *all* points equidistant from p and $\vec{r}(t_0)$ is the perpendicular bisector of the line segment joining p and $\vec{r}(t_0)$, shown as the dashed line k in Figure 1. The point we seek is the intersection of l and k.

What if l and k are parallel? In this special case, the line joining p and $\vec{r}(t_0)$ must be perpendicular to l, hence tangent to the directrix.

We have proved geometrically that for every point $\vec{r}(t_0)$ on the directrix there is exactly one point $\vec{q}(t_0)$ on the generalized parabola, except when the tangent to the directrix passes through p. Furthermore, we see that $\vec{q}$ can be parametrized with the same parameter as $\vec{r}$.

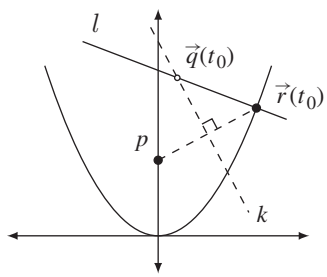


Figure 1. Defining $\vec{q}(t_0)$.

To find $\vec{q}$ analytically, the following must hold:

$$||\vec{q}(t_0) - p|| = ||\vec{q}(t_0) - \vec{r}(t_0)||,$$

which is the equidistant property, and

$$(\vec{q}(t_0) - \vec{r}(t_0)) \cdot \vec{r}\,'(t_0) = 0,$$

which ensures the vector pointing from $\vec{r}(t_0)$ to $\vec{q}(t_0)$ is perpendicular to $\vec{r}\,'(t)$. One can solve these equations for $x(t)$ and $y(t)$ by hand. Using *Mathematica*, we found

$$x(t) = \frac{2f(t)f'(t)(g(t) - y_0) + f(t)^2(-g'(t)) + g'(t)\left(-2y_0 g(t) + g(t)^2 + x_0^2 + y_0^2\right)}{2\left(f'(t)(g(t) - y_0) + (x_0 - f(t))g'(t)\right)} \tag{1}$$

$$y(t) = \frac{-x(t)f'(t) + f(t)f'(t) + g(t)g'(t)}{g'(t)}. \tag{2}$$

While the expression for $y(t)$ looks noticeably cleaner than the one for $x(t)$, it actually contains an $x(t)$ term, meaning neither is particularly illuminating. Regardless, $\vec{q}(t)$ can be computed and studied.

For example, with $p = (0, 1)$ and $\vec{r}(t) = \langle t, t^2\rangle$, equations (1) and (2) simplify to give:

$$\vec{q}(t) = \langle x(t), y(t)\rangle = \left\langle -\frac{t^3(t^2 - 2)}{1 + t^2}, \frac{3t^4 + t^2 + 1}{2t^2 + 2}\right\rangle.$$

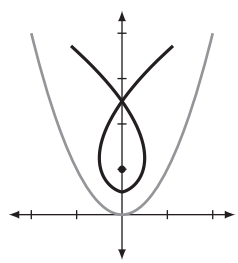
While the functions comprising $\vec{q}(t)$ do not seem simple, they allow us to do lots of exploring. Is $\vec{q}(t)$ defined everywhere? Is $\vec{q}(t)$ smooth everywhere? Where are horizontal and vertical tangent lines? What are the limits of $x(t)$ and $y(t)$ as t approaches infinity? We can arbitrarily choose functions for f and g, use a computer algebra system to plot $\vec{q}$, and see what kinds of parabolas are created.

A gallery of generalized parabolas

The `Manipulate` environment in *Mathematica* is an invaluable tool. We easily created an environment that plotted the parabola formed by a given directrix and moveable focus. One could move the focus around at will and immediately see the effect on $\vec{q}$. In the following figures, we give a taste of this.

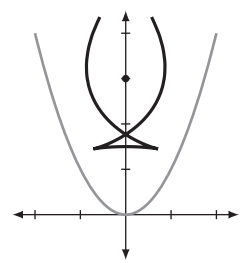
The first directrix the authors considered was a standard parabola, $\vec{r}(t) = \langle t, t^2 \rangle$. In Figure 2 we look at four placements of the focus which is represented by a filled black circle. Notice how, when the focus lies on the directrix's axis of symmetry, the parabola has the same symmetry. Also note the appearance of asymptotes and cusps. The geometric construction of $\vec{q}(t)$ given earlier explains where the asymptotes appear. We'll address cusps later.

At this point in the discovery process, we felt it was too soon to study the results analytically. Instead, we made the directrix a circle, parametrized by $\vec{r}(t) = \langle \cos t, \sin t \rangle$. Figure 3 indicates what we uncovered.



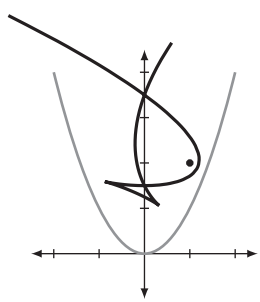
$p = (0, 1)$

$\vec{q}(t) = \left\langle -\frac{t^3(t^2-2)}{1+t^2}, \frac{3t^4+t^2+1}{2t^2+2} \right\rangle$



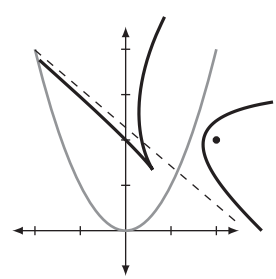
$p = (0, 3)$

$\vec{q}(t) = \left\langle -\frac{t(t^4-6t^2+6)}{t^2+3}, \frac{3t^4+t^2+9}{2t^2+6} \right\rangle$



$p = (1, 2)$

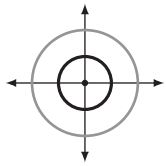
$\vec{q}(t) = \left\langle -\frac{t(t^4-4t^2+3)}{t^2-2t+2}, \frac{3t^4-4t^3+t^2-2t+5}{2t^2-4t+4} \right\rangle$



$p = (2, 2)$

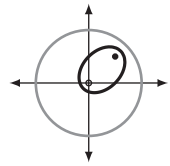
$\vec{q}(t) = \left\langle -\frac{t(t^4-4t^2+6)}{t^2-4t+2}, \frac{3t^4-8t^3+t^2-4t+8}{2t^2-8t+4} \right\rangle$

Figure 2. Parabolas with various foci and directrix $\vec{r}(t) = \langle t, t^2 \rangle$.



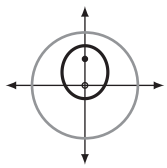
$p = (0, 0)$

$\vec{q}(t) = \left\langle \frac{1}{2}\cos t, \frac{1}{2}\sin t \right\rangle$



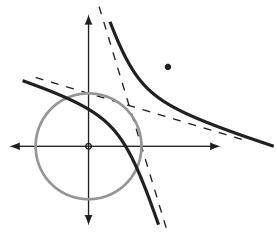
$p = (\frac{1}{2}, \frac{1}{2})$

$\vec{q}(t) = \left\langle -\frac{\cos t}{2\cos t + 2\sin t - 4}, -\frac{\sin t}{2\cos t + 2\sin t - 4} \right\rangle$



$p = (0, \frac{1}{2})$

$\vec{q}(t) = \left\langle \frac{3\cos t}{-4\sin t + 8}, \frac{3\sin t}{-4\sin t + 8} \right\rangle$



$p = (\frac{3}{2}, \frac{3}{2})$

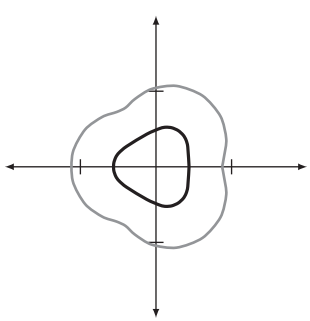
$\vec{q}(t) = \left\langle \frac{7\cos t}{6\cos t + 6\sin t - 4}, \frac{7\sin t}{6\cos t + 6\sin t - 4} \right\rangle$

Figure 3. Parabolas with various foci and directrix $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

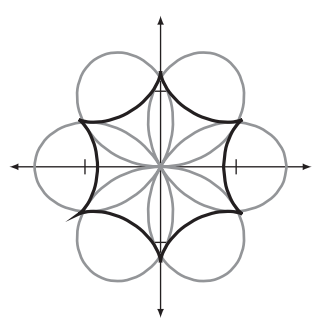
It is easy to see why, when $p = (0, 0)$, the parabola is a circle with the same center as the directrix and half the radius. In the other two graphs where the focus lies inside the directrix the parabola appears to be an ellipse. In fact, it is; one focus of the ellipse is p and the other is the origin (the center of the circle). When the focus lies outside a circle the resulting parabola is a hyperbola. These results are not new; see [**5**] for a nice proof and [**2**] for a visualization. In fact, with just a little thought one can sketch a proof by picture in the margins of this paper.

In Figures 4 through 7, the focus in each graph is the origin. Parametric equations for the directrices are given, but to avoid clutter we leave it to the reader to compute the equations of their parabolas.

As one looks at these graphs one gets a better feel for what the parabola for a given directrix will look like. For instance, one can accurately predict where asymptotes will appear. At the same time, surprises come often. For instance, consider the parabolas

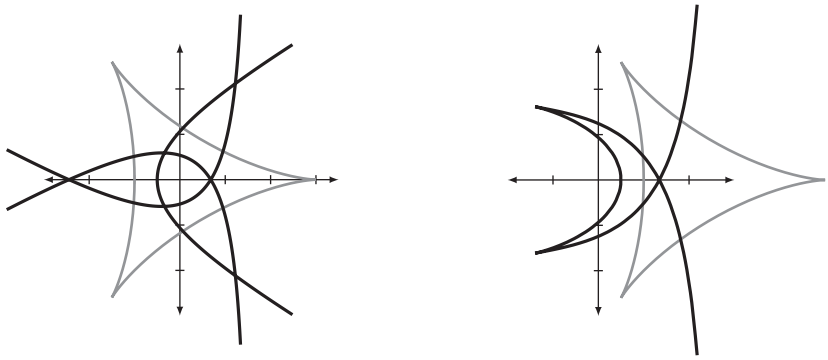


$\vec{r}(t) = \langle \cos t - \frac{1}{8}\cos(4t), \sin t - \frac{1}{8}\sin(4t) \rangle$



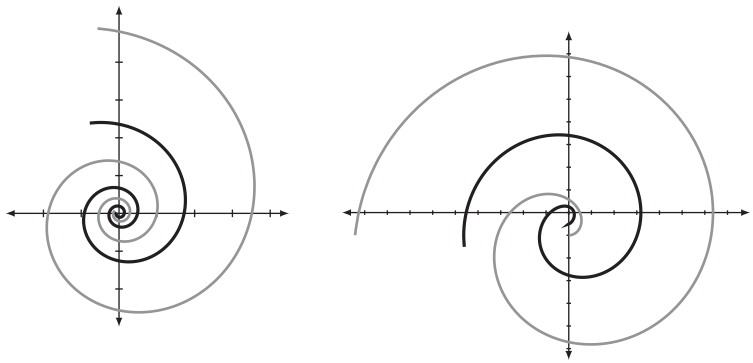
$\vec{r}(t) = \langle \frac{5}{6}(\cos t + \cos(5t)), \frac{5}{6}(\sin t - \sin(5t)) \rangle$

Figure 4. An epitrochoid and a spirograph.



$\vec{r}(t) = \langle 2\cos t + \cos(2t), 2\sin t - \sin(2t)\rangle$ $\vec{r}(t) = \langle 2\cos t + \cos(2t) + 2, 2\sin t - \sin(2t)\rangle$

Figure 5. Tricuspoids.



$\vec{r}(t) = \langle e^{5/t}\cos t, e^{t/5}\sin t\rangle$ $\vec{r}(t) = \langle t\cos t, t\sin t - 1\rangle$

Figure 6. Logarithmic and Archimedean spirals.

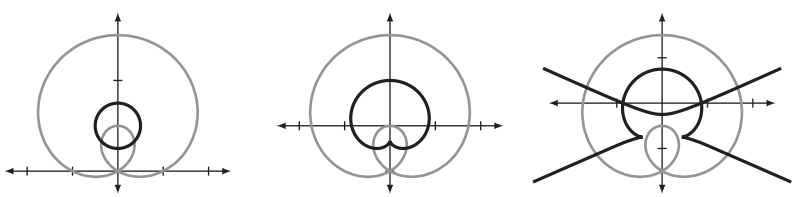


Figure 7. The limaçon $\vec{r}(t) = \langle \cos(t)(1 + 2\sin t), \sin(t)(1 + 2\sin t)\rangle$ with vertical shift of 0, -1, and $-3/2$ respectively.

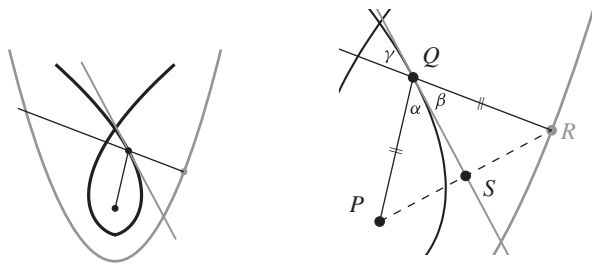
generated by the shifted limaçons in Figure 7. The parabola on the left is a circle and the parabola in the center is a cardiod.

Buoyed by the fact that, by accident, we had generated the conic sections, we began to think of the common properties of these shapes. One that is often taught in math courses is the reflection property. With the parabola, any ray perpendicular to the directrix and "inside" the parabola will reflect off the parabola toward the focus. Any ray emanating from one focus of an ellipse will reflect off the curve toward the other. Any ray going to one focus of a hyperbola will reflect off the curve toward the other focus. Are these reflection properties specific cases of a general principal, shared by

all generalized directrices and parabolas? That is the focus (pun intended) of the next section.

The Reflection Property

We begin by considering the directrix (in gray) and parabola (in black) in Figure 8(a). We add to this graph a line segment connecting the focus to $\vec{q}(t_0)$ (for some t_0), the line tangent to $\vec{q}(t_0)$ and the line perpendicular to $\vec{r}(t)$ at t_0. In Figure 8(b) we zoom in. We label the focus, $\vec{q}(t_0)$ and $\vec{r}(t_0)$ as P, Q and R respectively.



(a) (b)

Figure 8. The reflective property.

We wish to prove that $\alpha = \gamma$. Then a ray traveling along $\overleftrightarrow{PQ}$ will reflect off the parabola along $\overleftrightarrow{RQ}$. We do this below by showing $\alpha = \beta$.

Reflection Theorem. *Let $\vec{q}(t) = \langle x(t), y(t)\rangle$ be the (generalized) parabola with focus $P = (x_0, y_0)$ and directrix $\vec{r}(t) = \langle f(t), g(t)\rangle$. Let $\vec{r}(t)$ and $\vec{q}(t)$ be smooth at $t = t_0$. Set $R = \vec{r}(t_0)$ and $Q = \vec{q}(t_0)$. Let S be the point of intersection of the line tangent to $\vec{q}(t)$ at t_0 and $\overline{PR}$. Then $\angle PQS = \angle SQR$.*

Proof. Let $\alpha = \angle PQS$ and $\beta = \angle SQR$. We begin with the identity $||\vec{q}(t) - P|| = ||\vec{q}(t) - \vec{r}(t)||$, given by the construction of $\vec{q}(t)$. In terms of functions, we have

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - f)^2 + (y - g)^2}.$$

Squaring both sides and differentiating, one gets

$$(x - x_0)x' + (y - y_0)y' = (x - f)(x' - f') + (y - g)(y' - g'). \tag{3}$$

After a little algebra, we can rewrite this in vector notation as:

$$(\vec{r} - P) \cdot \vec{q}\,' = (\vec{r} - \vec{q}) \cdot \vec{r}\,'. \tag{4}$$

By construction, $\vec{r}(t) - \vec{q}(t)$ (i.e., $\overline{QR}$) is perpendicular to $\vec{r}\,'(t)$, hence the right hand side of (4) is 0. Therefore $\vec{q}\,'(t)$ is perpendicular to $\vec{r}(t) - P$ (i.e., $\overline{QS} \perp \overline{PR}$). Since the triangle $\triangle PQR$ is isosceles, this implies that the line tangent to $\vec{q}(t)$ at any $t = t_0$ bisects $\overline{PR}$ and the angle $\angle PQR$. Thus $\alpha = \beta$. ■

Conclusion

One can uncover a fascinating world by tweaking just one aspect of a well known concept. In our case, by generalizing the directrix we were able to produce a multitude of interesting and beautiful curves and generalize the reflection property of conic sections.

The curves developed here are a type of *derived* curve, that is, they are derived from another curve (or curves) through a systematic process. There is no exhaustive list of all types of derived curves nor is there an objective standard for which is most important in some sense, but searching the literature demonstrates that certain curves get more attention than others. (For overviews of famous curves, we suggest [**3**], [**4**], [**7**] and [**8**]; *MathWorld* [**9**] is another wonderful resource.) Of all derived curves, one could argue that the evolute and involute (inverse constructions) are the most famous. We briefly describe the evolute here.

Given a smooth curve C, an *osculating circle* is a circle tangent to C with curvature equal to the curvature of C at the point of tangency. The *evolute* of C is the locus of centers of all osculating circles. Our generalized parabolas could have been defined as the locus of centers of circles that are tangent to C (the directrix) and contain a certain fixed point (the focus). Thus the generalized parabola is similar in spirit to the evolute.

Our construction appears to be relatively unexplored. In [**6**], Pedoe refers to it as a *circle tangent curve* and produces ellipses and hyperbolas from circles and points. Generalized parabolas also appear in [**1**] though the choice of directrix is limited to conic sections, and the authors actually tackle the more difficult problem of finding *superbolas*, which are the locus of points equidistant from two nonintersecting conics. The Reflection Theorem does not seem to appear anywhere in the literature.

Challenges

To encourage readers to explore generalized parabolas further, we offer some problems.

1. Formulas (1) and (2) are not very practical. As in Figures 4–7, we always place the focus at the origin. This simplifies these expressions, but further simplification is possible. One way is to write $\vec{q}$ as the sum of two vectors, one parallel to $\vec{r}$ and the other perpendicular to $\vec{r}\,'$; that is,

$$\vec{q}(t) = \vec{r}(t) - \frac{\vec{r}(t)\cdot\vec{r}(t)}{2\vec{r}(t)\cdot\vec{r}'_{\perp}(t)}\vec{r}'_{\perp}(t), \tag{5}$$

   where $\vec{r}'_{\perp}(t)$ is the vector $\vec{r}\,'(t)$ rotated counter-clockwise by 90°. The first challenge is to derive equation (5) and then go on to write $\vec{q}(t)$ as the sum of a vector parallel to $\vec{r}(t)$ and a vector perpendicular to $\vec{r}(t)$.
2. The appearance of cusps in generalized parabolas is related to the curvature of the directrix. We challenge the reader to discover how.
3. We assume the focus is not on the directrix. The reader may have noticed that some of our images violate this assumption suggesting that things can work fine, but what issues need to be considered and how can they be resolved?
4. What curves result if we repeat the parabola generating process? Specifically, given $\vec{r}(t)$ and p, compute $\vec{q}(t)$, then use p and $\vec{q}(t)$ as another focus/directrix pair. What new parabola is thus created, and what is its relationship to $\vec{r}(t)$? If we

iterate, what can be said about the set of parabolas formed? Can they converge pointwise or uniformly to a curve?

5. Does a directrix exist where the set of all iteratively generated parabolas is finite? One of the examples in the gallery gives hope that this is possible.
6. Can this construction be extended into higher dimensions? A paraboloid is the set of all points equidistant from a point and a plane. What surfaces are generated if the plane is replaced by a curved surface? What surface will generate an ellipsoid?
7. Show that the parabola of a logarithmic spiral is a congruent logarithmic spiral.
8. Find an inverse process: start with any curve and find a focus–directrix pair (if one exists) that generates it.
9. Applications exist of the reflection property of conics. Find an application of the reflection property of a non-conic, generalized parabola.

Summary. In this article we explore the consequences of modifying the common definition of a parabola by considering the locus of all points equidistant from a focus and (not necessarily linear) directrix. The resulting derived curves, which we call *generalized parabolas*, are often quite beautiful and possess many interesting properties. We show that the conic sections are generalized parabolas whose directrix is a circle or a line, and their well known reflective properties are actually specific instances of a reflective property held by all generalized parabolas. Finally, we offer the reader suggestions for further investigation.

References

1. D. A. Field and R. Field, A new family of curves for industrial applications, Technical Report GMR-7571, General Motors Research Laboratories, 1992.
2. G. Hartman, Steiner chain of circles (2009); available at `http://demonstrations.wolfram.com/SteinerChainOfCircles/`.
3. J. D. Lawrence, *A Catalog of Special Plane Curves*, Dover Press, New York, 1972.
4. E. H. Lockwood, *A Book of Curves*, Cambridge University Press, New York, 1961.
5. C. S. Ogilvy, *Excursions in Geometry*, Oxford University Press, New York, 1969.
6. D. Pedoe, *Circles*, vol. 2, *International Series of Monographs in Pure and Applied Mathematics*, Pergamon Press, New York, 1957.
7. E.V. Shikin, *Handbook and Atlas of Curves*, CRC Press, Boca Raton, 1995.
8. J. Wassenaar, Mathematical curves (2010); available at `http://www.2dcurves.com`.
9. E. Weisstein, Plane Curve—From MathWorld; available at `http://mathworld.wolfram.com/PlaneCurve.html`.