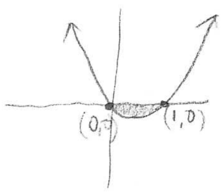
Exam 1 Calc 2 2/7/2014 [redacted]

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an integral for the area of the region bounded between $y = x^2 - x$ and the x-axis.



$$\boxed{\int_0^1 (-x^2 + x)\,dx}$$

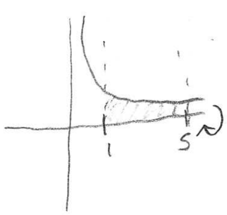
Excellent

2. Let $L(x) = \int_1^x \frac{1}{t}\,dt$. What is $L'(x)$?

$\boxed{L'(x) = \frac{1}{x}}$, This is because of the Fundamental Theorem of Calculus Part 1!

Great!

3. Suppose the region between $y = \dfrac{1}{x}$ and the x-axis on the interval from $x = 1$ to $x = 5$ is rotated around the x-axis. Set up an integral for the volume of the solid produced.



$$\pi \int_1^5 \left(\frac{1}{x}\right)^2 - 0^2\,dx$$

$$\boxed{\pi \int_1^5 \frac{1}{x^2}\,dx}$$

$\left(\frac{1}{x}\right)\left(\frac{1}{x}\right) = \frac{1}{x^2}$

Great!

4. If a spring has a natural length of 30cm, and 50N of force is required to hold it stretched to 35cm, how much work would be required to stretch it from 35cm to 45cm?

Hooke's: $f = kx$ so $(50) = k\underbrace{(0.05)}_{\text{35cm is 0.05m beyond natural length}} \Rightarrow k = 1000\ N/m$

Then work for 0.05m to 0.15m beyond natural length is:

$$W = \int_{0.05}^{0.15} 1000x\,dx$$

$$= 500x^2 \Big|_{0.05}^{0.15}$$

$$= 11.25 - 1.25$$

$$= 10\ J$$

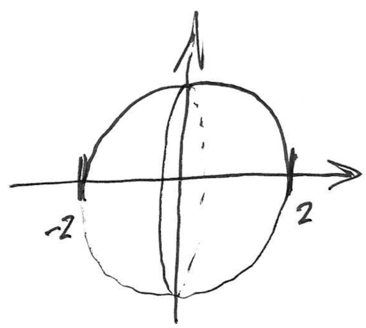
5. Set up an integral for the area of the surface produced by rotating the curve $y=\sqrt{4-x^2}$ around the x-axis.

$$\boxed{\text{Area} = \int_a^b 2\pi \cdot f(x) \cdot \sqrt{1+[f'(x)]^2}\, dx}$$

So for $y = (4-x^2)^{1/2}$

$$y' = \tfrac{1}{2}(4-x^2)^{-1/2} \cdot -2x$$

$$= \frac{-x}{\sqrt{4-x^2}}$$



Then area $= \displaystyle\int_{-2}^{2} 2\pi\sqrt{4-x^2} \cdot \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$

or $\displaystyle\int_{-2}^{2} 2\pi\sqrt{4-x^2}\, \sqrt{1+\frac{x^2}{4-x^2}}\, dx$

6. Find the length of the curve $y = x^{3/2}$ between $x = 0$ and $x = 4$.

$$y' = \frac{3}{2}x^{1/2}$$

$$\text{Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{4 + 9x}\, dx$$

Let $u = 4 + 9x$

$$\frac{du}{dx} = 9$$

$$\frac{du}{9} = dx$$

$$= \frac{1}{2}\int_{x=0}^{x=4} u^{1/2}\, dx$$

$$= \frac{1}{2}\frac{1}{9}\frac{2}{3}u^{3/2}\Big|_{x=0}^{x=4}$$

$$= \frac{1}{27}(4 + 9x)^{3/2}\Big|_0^4$$

$$= \frac{1}{27}(40)^{3/2} - \frac{1}{27}(4)^{3/2}$$

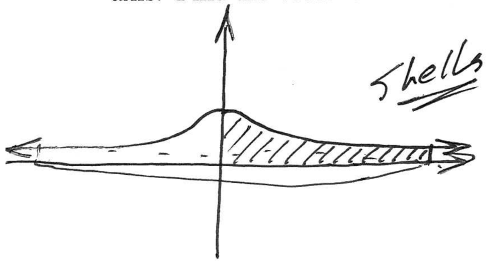
7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Whoa! Calc 1 was okay, but now in Calc 2 the professor keeps saying stuff like 'If you think about it, it's clear...' when dude, it's not clear to me. I'm pretty good at working stuff out, but this thinking stuff just isn't why I take math. Yesterday he said if we think about it, we already know what the answer's supposed to be if you rotate the top half of a circle with its center at the origin around the *x*-axis to do its volume. Dude, I can probably work it out, but how am I supposed to already know the answer? Was it in the back of the book or something? Totally unfair!"

   Help Biff out. Explain to him as clearly as possible why he really should already know what the answer to this problem is.

Biff probably took geometry in highschool So he should be able to recognize $\frac{4}{3}\pi r^3$ as the volume of a sphere. So if Biff makes a picture of the semi-circle and revolves it he'll see it creates a sphere, an object he should remember how to take the volume of from highschool.

Great!

8. The first-quadrant region under the curve $y = \dfrac{1}{1+x^2}$ but left of $x = 5$ is rotated around the y-axis. Find the volume of the resulting solid.



Shells!

$$V = \int_0^5 2\pi \underbrace{x}_{\text{radius}} \underbrace{\left(\frac{1}{1+x^2}\right)}_{\text{height}} dx$$

$$= \int_{x=0}^{x=5} 2\pi \cancel{x} \cdot \frac{1}{u} \cdot \frac{du}{2\cancel{x}}$$

Let $u = 1 + x^2$

$\dfrac{du}{dx} = 2x$

$\dfrac{du}{2x} = dx$

$$= \int_{x=0}^{x=5} 2\pi \cdot \frac{1}{2} \cdot \frac{1}{u}\, du$$

$$= \pi \cdot \ln|u| \Big|_{x=0}^{x=5}$$

$$= \pi \ln(1+x^2) \Big|_0^5$$

$$= \pi \ln 26 - \pi \ln 1$$

$$= \pi \ln 26$$

9. Derive the integration formula $\int x\sqrt{ax+b}\,dx = \dfrac{2}{15a^2}(3ax-2b)(ax+b)^{3/2}+C$.

$$\int x\sqrt{ax+b}\,dx$$

$u = ax+b\,dx$

$du = a\,dx$

$dx = \frac{1}{a}du$

$ax = u-b$

$x = \frac{u-b}{a}$

$$\frac{1}{a}\int \sqrt{u}\left(\frac{u-b}{a}\right)du$$

$$\frac{1}{a^2}\int \sqrt{u}(u-b)\,du$$

$$\frac{1}{a^2}\int u^{\frac{3}{2}} - bu^{\frac{1}{2}}\,du$$

$$\frac{1}{a^2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2b}{3}u^{\frac{3}{2}}\right) + C$$

$$\frac{1}{a^2}\left(\frac{2}{5}(ax+b)^{\frac{5}{2}} - \frac{2}{3}b(ax+b)^{\frac{3}{2}}\right)$$

$$\frac{1}{a^2}\left(\frac{6}{15}(ax+b)^{\frac{5}{2}} - \frac{10b}{15}(ax+b)^{\frac{3}{2}}\right)$$

$$\frac{2}{15a^2}\left(3(ax+b)^{\frac{5}{2}} - 5b(ax+b)^{\frac{3}{2}}\right)$$

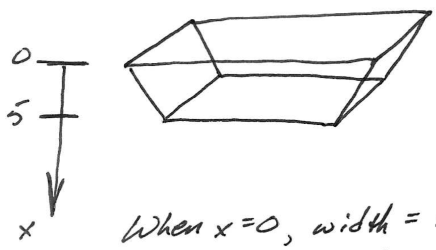
$$\frac{2}{15a^2}(ax+b)^{\frac{3}{2}}\left(3(ax+b) - 5b\right) + C$$

$$\frac{2}{15a^2}(ax+b)^{\frac{3}{2}}\left(3ax+3b-5b\right) + C$$

$$\frac{2}{15a^2}(ax+b)^{\frac{3}{2}}\left(3ax-2b\right) + C$$

Well done

10. Jon plans to dig a ginormous trench in the center of Coe's quad. The trench will be 5 feet deep and a 5 foot by 20 foot rectangle at the bottom, with sloped sides so that at the top it's a 15 foot by 30 foot rectangle. The trench will then be filled with Jello (which has a density of 71.3 lbs. /ft.^{3}). Set up an integral for the amount of work required to pump all of the Jello out of the trench.



Area of a Slice $= (15-2x)(30-2x)\ ft^2$

Volume " " " $= (15-2x)(30-2x)\,\Delta x\ ft^3$

Force " " " $= (15-2x)(30-2x)\cdot 71.3\,\frac{lbs}{ft^3}\cdot ft^3\,\Delta x$

Work " " " $= (15-2x)(30-2x)\cdot 71.3\cdot x\,\Delta x\ ft\cdot lbs.$

When $x=0$, width $= 15$
$x=5$, width $= 5$

When $x=0$, length $= 30$
$x=5$, length $= 20$

Total Work $= \int_0^5 (15-2x)(30-2x)\cdot 71.3x\,dx$

width $= 15-2x$
length $= 30-2x$