Exam 3 Calc 2 4/4/2014

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write the 3rd degree Maclaurin polynomial for e^x.

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$$

Great

2. Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ Which is a P-series with

$\boxed{P \leq 1 \text{ so } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}}$

Excellent!

3. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges or diverges.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ Alternating Series Test

— $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

— derivative of $\frac{1}{\sqrt{n}} = -\frac{1}{2} n^{-\frac{3}{2}}$ Because the derivative is negative, it is decreasing.

Therefore by the A.S.T. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

Great

4. Determine whether the series $\sum_{k=1}^{\infty} \frac{k^2 - 1}{k^3 + 4}$ converges or diverges.

Limit Comparison to $\sum \frac{1}{k}$, the harmonic series, which diverges

$$\lim_{k\to\infty} \frac{\frac{k^2-1}{k^3+4}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2-1}{k^3+4} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{k^3-k}{k^3+4}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^3}}{\cancel{k^3}} \cdot \frac{1-\frac{1}{k^2}}{1+\frac{4}{k^3}}$$

$$= \frac{1}{1} = 1$$

So since that limit is finite and non-zero, both series do the same thing, and this one must also diverge.

5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

Integral Test!

Let $u = \ln x$ $\Rightarrow x\,du = dx$

$\frac{du}{dx} = \frac{1}{x}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{1}{\cancel{x}\cdot u^2}\cdot \cancel{x}\,du$$

$$= \lim_{b\to\infty} \int_{x=2}^{x=b} u^{-2}\,du = \lim_{b\to\infty} \left. \frac{u^{-1}}{-1} \right|_{x=2}^{x=b} = \lim_{b\to\infty} \frac{-1}{\ln b} - \frac{-1}{\ln 2} = 0 + \frac{1}{\ln 2}$$

So since that integral converges, the series also converges.

6. Determine the radius of convergence of the power series $\sum \left(\frac{x}{3}\right)^k$.

Ratio Test!

$$\lim_{k\to\infty} \left| \frac{\left(\frac{x}{3}\right)^{k+1}}{\left(\frac{x}{3}\right)^k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{3^{k+1}} \cdot \frac{3^k}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x}{3} \right|$$

So the Ratio Test promises it converges provided $\left|\frac{x}{3}\right| < 1$, or $\frac{|x|}{3} < 1$, or $|x| < 3$, which means the radius of convergence is 3.

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Well, crap. I'm getting okay at finding these Taylor series and stuff, 'cause I found there's a formula in the book. But then there's all these other things they bring in and I'm pretty lost. I might have to kill my roommate, 'cause they say you get all A's for a semester if your roommate dies. But if I can figure stuff out by the exam tomorrow, I guess I won't have to do that. So like one of the things the prof said we needed to know was why the series x to the n sums up to 1 over 1 minus x, and he said it was more an explaining thing about reasons than a bunch of calculating, but I'm not so good with reasons. Maybe I need to think more about the roommate option..."

Help Biff (and his roommate!) by explaining clearly how we can find the sum of $\sum x^n$.

The geometric series test states that $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$ iff $|r|<1$.

$\sum_{n=0}^{\infty} x^n$ has an a of 1 and an r of x, so

$\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ when $|x|<1$,

Which means the sum is $\frac{1}{1-x}$ when $-1<x<1$.

- Excellent

8. Use a Taylor series with at least 4 nonzero terms to approximate $\sqrt{e}$.

I know that $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\sqrt{e} = e^{1/2}$ so $x = \frac{1}{2}$

that means $e^{1/2} \approx 1 + \frac{1}{2} + \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{6}$

$\boxed{\text{so } \sqrt{e} \approx 1.646}$

Excellent.

9. Use a Taylor series with at least 3 nonzero terms to approximate $\int_0^{0.2} \sin(x^2)\,dx$.

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin(x^2) \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}$

$\int_0^{.2} \sin(x^2)\,dx \approx \int_0^{.2} x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10}\,dx$

$\left[\frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11}\right]_0^{.2}$

$\boxed{\frac{1}{3}(.2)^3 - \frac{1}{42}(.2)^7 + \frac{1}{1320}(.2)^{11}} \approx .0026663619$

Well done.

no specified degree

10. Use a Taylor series to evaluate $\lim_{x\to 0} \frac{x}{e^x - e^{-x}}$

I know $e^x \approx 1+x$

so $e^{(-x)} \approx 1-x$

so $\lim_{x\to 0} \frac{x}{e^x - e^{-x}} = \lim_{x\to 0} \frac{x}{1+x-1+x}$

$= \lim_{x\to 0} \frac{x}{2x} \overset{L'H}{=} \frac{1}{2}$

$\boxed{\text{So } \lim_{x\to 0} \frac{x}{e^x - e^{-x}} \approx \frac{1}{2}}$

Nice Job.