Exam 1 Differential Equations 2/14/14

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Is $y(t) = t^2 - 2t + 2 + 5e^{-t}$ a solution to the differential equation $\frac{dy}{dt} = -y + t^2$?

$y'(t) = 2t - 2 - 5e^{-t}$

$2t - 2 - 5e^{-t} = -\cancel{t^2} + 2t - 2 - 5e^{-t} + \cancel{t^2}$

$2t - 2 - 5e^{-t} = 2t - 2 - 5e^{-t}$

Since we plugged it in and it worked, yes, this is a solution.

Great

2. Find a general solution to the differential equation $\frac{dy}{dt} = ty$.

$\int \frac{dy}{y} = \int \frac{t\cancel{y}}{\cancel{y}}\,dt$

$\ln|y| = \frac{t^2}{2} + C$

$e^{\ln y} = Ae^{t^2/2}$ (abs. val. taken care of by A)

$\boxed{y = Ae^{t^2/2}}$

Great!

3. Sketch a phase line for the differential equation $\frac{dP}{dt} = 0.0037P(4000 - P)$.

$0 = 0.0037P(4000-P)$

$0.0037P = 0$ and $4000-P=0$

so there are equilibrium points at $\underline{P=0}$ and $\underline{P=4000}$

Excellent!

$(-1): (-)(+) = - < 0$

$(1): (+)(+) = + > 0$

$(4001): (+)(-) = - < 0$

$(y-4t)^2$

4. Consider the differential equation $\frac{dy}{dt} = y - 4t + y^2 - 8yt + 16t^2 + 4$. Change the dependent variable from y to u using the change of variables $u = y - 4t$.

$\frac{du}{dt} = \frac{dy}{dt} - 4$

$\frac{du}{dt} + \cancel{4} = u + u^2 \cancel{+4}$

$\frac{du}{dt} = u + u^2$

Good.

5. Consider the differential equation $\dfrac{dT}{dt} = 0.02(30 - T)$, with initial condition $T(0) = 170$, modeling the temperature of a cup of coffee sitting in a car's cupholder. Use Euler's method with $\Delta t = 5$ to approximate the temperature of the coffee after 10 minutes, when the driver spills it on his pants just as he arrives at his Valentine's date.

t	dT/dt	T
0	-2.8	170
5	-2.52	156
10	-2.268	143.4

$T_{k+1} = T_k + f\Delta t$

$\dfrac{dT}{dt} = f$

$\left.\dfrac{dT}{dt}\right|_{T=170} = -2.8$

$170 + -2.8 \cdot 5 = 156$ Good

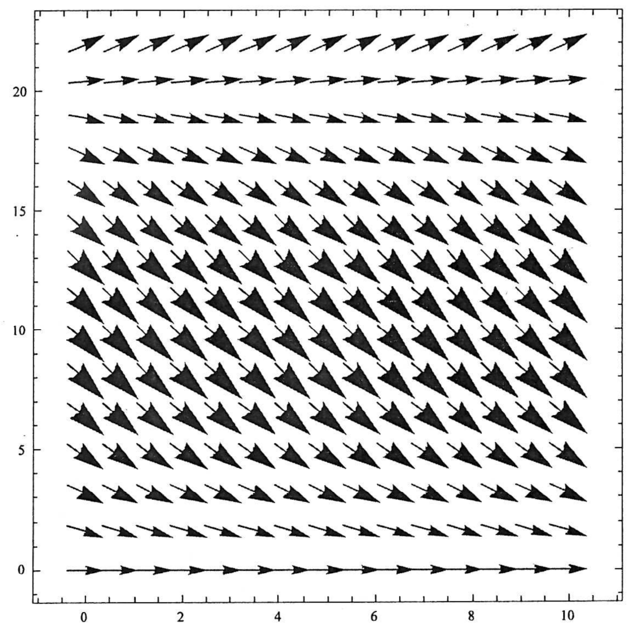
$\left.\dfrac{dT}{dt}\right|_{T=156} = -2.52$

$156 + -2.52 \cdot 5 = 143.4$

$T(10) \approx 143.4$

6. Bunny is taking Differential Equations at Enormous State University, and is having some trouble. Bunny says "Ohmygod! We're supposed to use computers for this class, and I'm so totally lost! I turned this slope thingy in for this one where we were supposed to make it, like, logistic population, right? So where there's carry capacity and stuff? And the grader told me no credit because it's obviously wrong, but I typed it exactly like it was in the book. How can he tell it's wrong anyway?"

   Explain clearly to Bunny which characteristics of this slope field fit a logistic growth model, and which do not.



A plot of a logistic growth model would expect to look like

where the population growth rate decreases as it reaches the carrying capacity. We would expect the arrows to be pointing up between 0 and the carrying capacity, and then downward above the carrying capacity since a population cannot sustain itself above that level due to limitations. Bunny probably typed in a sign wrong somewhere.

Excellent!

7. Find the power series expansion for the general solution up to degree four to the differential equation $\frac{dy}{dt} = -2ty$.

$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$

$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3$

$-2ty = -2a_0 t - 2a_1 t^2 - 2a_2 t^3$

$a_1 = 0$

$2a_2 = -2a_0 \quad a_2 = -a_0$

Excellent!

$3a_3 = -2a_1 = 0 \quad a_3 = 0$

$4a_4 = -2a_2 = -2 \cdot -a_0 = 2a_0 \quad a_4 = \frac{a_0}{2}$

$y = a_0 - a_0 t^2 + \frac{a_0}{2} t^4$

7. Find the power series expansion for the general solution up to degree four to the differential equation $\frac{dy}{dt} = -2ty$.

let $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$

$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$

$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 = -2t\left[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5\right]$

$= -2a_0 t - 2a_1 t^2 - 2a_2 t^3 - 2a_3 t^4 - 2a_4 t^5 - 2a_5 t^6$

t

t^0: $a_1 = 0$

t^1: $2a_2 = -2a_0 \Rightarrow a_2 = -a_0$

t^2: $3a_3 = -2a_1 \Rightarrow a_3 = -\frac{2}{3}a_1 \Rightarrow a_3 = 0$

t^3: $4a_4 = -2a_2 \Rightarrow a_4 = -\frac{1}{2}a_2 \Rightarrow a_4 = \frac{1}{2}a_0$

t^4: $5a_5 = -2a_3 \Rightarrow a_5 = -\frac{2}{5}a_3 \Rightarrow a_5 = 0$

$y(t) = a_0 + \cancel{0t} + -a_0 t + \cancel{0t^3} + \frac{1}{2}a_0 t^4$

$y(t) = a_0 - a_0 t + \frac{1}{2}a_0 t^4$

Excellent

linear

8. Find a general solution to the differential equation $\frac{dx}{dt} - 2x = 3k_2 e^{-4t}$.

$$\frac{dx}{dt} - 2x = 3k_2 e^{-4t}$$

$$\text{let } \mu = e^{\int -2\, dt} = e^{-2t}$$

$$\frac{dx}{dt} e^{-2t} - 2x e^{-2t} = 3k_2 e^{-4t} e^{-2t}$$

$$\int \frac{d}{dt}\left(x e^{-2t}\right) = \int 3k_2 e^{-6t}$$

$$x e^{-2t} = 3k_2 (-1/6) e^{-6t} + C$$

$$x = \frac{-\frac{1}{2} k_2 e^{-6t} + C}{e^{-2t}}$$

$$\underline{x = -\frac{1}{2} k_2 e^{-4t} + C e^{2t}}$$

Well done!

general solution

9. **Sketch the bifurcation diagram** for the differential equation $\dfrac{dy}{dt} = y^2 - 6y + \mu$. Include direction arrows on the phase lines and make clear the exact μ values where bifurcations occur.

Look at equilibra, so where $0 = y^2 - 6y + \mu$

The quadratic formula says $y = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(\mu)}}{2(1)}$

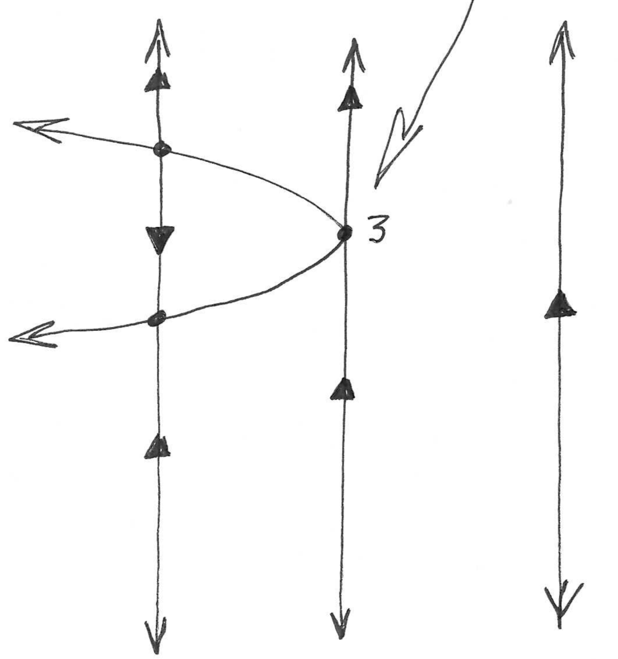
and the discriminant of that is $36 - 4\mu$, so

$\mu = 9$ means one root,

$\mu > 9$ means no roots

$\mu < 9$ means two roots

Look! A bifurcation!



$\mu < 9$ $\qquad$ $\mu = 9$ $\qquad$ $\mu > 9$

equilibrium $y = \dfrac{6 \pm \sqrt{36 - 4\mu}}{2}$

equilibrium $y = \dfrac{6 \pm \sqrt{0}}{2}$

10. The bunnies on Valentine's Island have an unusual genetic variation that gives them all pink fur. When tourism begins on the island, the bunny population is 30,000. The bunny population grows 6% each year through natural reproduction. Tourists smuggle bunnies off the island, first at a modest rate of 100 bunnies per year in year 1, but then growing by 100 bunnies each year, so that by year 10, there are 1000 bunnies smuggled off the island, and so forth.

a) Write a differential equation for the rabbit population on Valentine's Island.
b) Find a general solution to this differential equation.
c) Find the particular solution satisfying this initial condition.

a) $\frac{db}{dt} = .06b - 100t$

$\frac{db}{dt} - .06b = -100t$

It's linear! $\mu(t) = e^{\int -.06dt} = e^{-.06t}$

$\frac{db}{dt} \cdot e^{-.06t} - .06e^{-.06t}b = -100t \cdot e^{-.06t}$

$\frac{d}{dt}\left(b \cdot e^{-.06t}\right) = -100t \cdot e^{-.06t}$

$\int t \cdot e^{-.06t} dt$

$u = t \qquad v = \frac{-1}{.06} e^{-.06t}$

$u' = 1 \qquad v' = e^{-.06t}$

$b \cdot e^{-.06t} = -100\left(\frac{-1}{.06} t e^{-.06t} - \int \frac{-1}{.06} e^{-.06t} dt\right)$

$= -100\left(\frac{-1}{.06} t e^{-.06t} + \frac{-1}{.06^2} e^{-.06t}\right) + C$

b) $b = \frac{100}{.06} t + \frac{100}{.06^2} + C \cdot e^{.06t}$

Since $P(0) = 30{,}000$: $30{,}000 = 1666.\overline{6} \cdot 0 + 27{,}777.\overline{7} + C \cdot e^0$

$2{,}222.\overline{2} = C$

c) $b = 1666.\overline{6}t + 27{,}777.\overline{7} + 2222.\overline{2}e^{.06t}$