Examlet 1b　　Foundations of Advanced Math　　2/6/14

1. Show that the square of a throddodd integer is throdd.

Proof: Let's say $x \in \mathbb{Z}$ is our throddodd integer, so $x = 3a+2$, for $a \in \mathbb{Z}$.

$x^2 = (3a+2)^2$

$x^2 = 9a^2+12a+4$

$x^2 = 3(3a^2+4a+1)+1$

Since $(3a^2+4a+1) \in \mathbb{Z}$ (by C.O.I.),

we can conclude that x^2 is throdd,

and that the square of a throddodd integer is throdd. $\square$

Great

2. Determine whether $(P \wedge Q) \Rightarrow R$ is logically equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.

P	Q	R	$P \wedge Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \wedge Q) \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

No, $(P \wedge Q) = R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are not logically equivalent because the truth values in those two columns are not identical.

Good

3. If $a \equiv_n b$, and $b \equiv_n c$ then $a \equiv_n c$.

If $a \equiv_n b$ then $n \mid b-a$ so $b-a=nx$ for $x \in \mathbb{Z}$.

If $b \equiv_n c$ then $n \mid c-b$ so $c-b=ny$ for $y \in \mathbb{Z}$.

$b-a=nx$

Also, $b=nx+a$

we can substitute

$c-(nx+a)=ny$

$c=ny+nx+a$

$c-a=n(y+x)$

$y+x \in \mathbb{Z}$ by COI, so $n \mid c-a$ and we can conclude $a \equiv_n c$. $\square$

Well done.

4. $\sqrt{2}$ is irrational.

Suppose $\sqrt{2}$ was rational, so $\sqrt{2} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ reduced so there are no common factors.
Then squaring both sides gives $2 = \frac{p^2}{q^2} \Leftrightarrow 2q^2 = p^2$. The left side of the equation is even by definition because $q^2 \in \mathbb{Z}$ by C.O.I. This would mean p^2 would have to be even, and as previously proven, p would also be even, so $p = 2n$ for $n \in \mathbb{Z}$.
Then $2q^2 = (2n)^2 \Leftrightarrow 2q^2 = 4n^2 \Leftrightarrow 2q^2 = 2(2n^2) \Leftrightarrow q^2 = 2n^2$ which means that q^2 and thus q would both be even as well. Since this contradicts our original statement of p and q having no common factors, this clearly can't be true, therefore $\sqrt{2}$ must be irrational. $\square$

Well done!

5. For all $n \in \mathbb{N}$, $n \leq 2^n$.

Let's induct!

First a base case, so let $n=0$. Then our statement is $(0) \leq 2^{(0)}$,

or $0 \leq 1$, which we think is true.

Then s'pose it's true for some $n = k \in \mathbb{N}$, so $(k) \leq 2^{(k)}$.

Now we look at the statement for $n = k+1$, which is $k+1 \overset{?}{\leq} 2^{(k+1)}$.

We know by inductive hypothesis that $k \leq 2^k$. If we also use the fact*

that $1 \leq 2^k$ for $k \in \mathbb{N}$, and add these two inequalities, we have

$k+1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ as desired.

So since $n \leq 2^n$ is true for $n=0$, and when it's true for $k \in \mathbb{N}$ it's also true for $k+1$, then by mathematical induction it's true for all $n \in \mathbb{N}$. □

* <u>Prop.</u>: $\forall n \in \mathbb{N}$, $1 \leq 2^n$

<u>Proof</u>: Base case is $1 \leq 2^0$, which is true.

Then s'pose it's true for $n = k \in \mathbb{N}$, so $1 \leq 2^k$. Multiply both sides by 2 to get $2 \leq 2^{k+1}$, and since $1 \leq 2$ we have $1 \leq 2 \leq 2^{k+1}$, so the statement holds for $k+1$ as well.

Thus since it's true for $n=0$, and when it's true for $k \in \mathbb{N}$ it's also true for $k+1$, then by mathematical induction it's true for all $n \in \mathbb{N}$. □