6. Use Cardano's technique (of substituting $z = x - \frac{a_2}{3}$) to solve the following cubics.

(a) $z^3 - 6z^2 - 3z + 18 = 0$.

(b) $z^3 + 3z^2 - 24z + 28 = 0$.

7. Refer to Figure 1.6(a). The two factor vectors are $\mathbf{a} = (\mathbf{2}, \mathbf{1})$ and $\mathbf{b} = (\mathbf{1}, \mathbf{3})$.

(a) Find the length of vectors **a** and **b**.

(b) Using your calculator, compute the radian and degree measure of angles α and β.

(c) Using Wessel's rules for vector multiplication find:

i. The length of the product vector **c**.

ii. The radian and degree measure of the angular displacement of the product vector **c**.

(d) Using your calculator, get the coordinate representation of the product vector **c**.

(Note: You will learn a slicker technique for these computations in Section 1.2.)

8. Explain why it would have been a bad idea for Wessel to stipulate that the angular displacement of the product of two vectors equaled the product of the displacements of the the two vectors.
Hint: What would be the result of multiplying the vector **i** with the standard unit vector? What would the product $(-\mathbf{1})(-\mathbf{1})$ equal? Finally, show that it would be possible to have non-zero vectors satisfying $\mathbf{ab} = \mathbf{ac}$, but $\mathbf{b} \neq \mathbf{c}$.

9. Write a paper that compares Wallis' representation of complex numbers with the procedure outlined in the article by Alec Norton and Benjamin Lotto: "Complex Roots Made Visible," *The College Mathematics Journal*, 15(3), June 1984, pp. 248–249.

10. Investigate library and/or web resources and write up a detailed analysis explaining why the solution to the depressed cubic, Equation (1-3), is valid. *Hint*: A good reference is the article by Dan Kalman and James White: "A Simple Solution of the Cubic," *The College Mathematics Journal*, 29(5), November 1998, pp. 415–418.

1.2 THE ALGEBRA OF COMPLEX NUMBERS

We have shown that complex numbers came to be viewed as ordered pairs of real numbers. That is, a complex number z is defined to be

$$z = (x,\ y), \tag{1-7}$$

where x and y are both real numbers.

The reason we say *ordered* pair is because we are thinking of a point in the plane. The point (2, 3), for example, is not the same as (3, 2). The *order* in which we write x and y in Equation (1-7) makes a difference. Clearly, then, two complex numbers are equal if and only if their x coordinates are equal *and* their y coordinates are equal. In other words,

$$(x,\ y) = (u,\ v) \text{ iff } \quad x = u \quad \textit{and} \quad y = v.$$

(Throughout this text, iff means *if and only if*.)

A meaningful number system requires a method for combining ordered pairs. The definition of algebraic operations must be consistent so that the sum, difference, product, and quotient of any two ordered pairs will again be an ordered pair. The key to defining how these numbers should be manipulated is to follow Gauss's lead and equate $(x,\ y)$ with $x + iy$. Then, if $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$ are arbitrary complex numbers, we have

$$\begin{aligned} z_1 + z_2 &= (x_1,\ y_1) + (x_2,\ y_2) \\ &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i\,(y_1 + y_2) \\ &= (x_1 + x_2,\ y_1 + y_2)\,. \end{aligned}$$

Thus, the following definitions should make sense.

Definition 1.1: Addition

$$\begin{aligned} z_1 + z_2 &= (x_1,\ y_1) + (x_2,\ y_2) \\ &= (x_1 + x_2,\ y_1 + y_2)\,. \end{aligned} \tag{1-8}$$

Definition 1.2: Subtraction

$$\begin{aligned} z_1 - z_2 &= (x_1,\ y_1) - (x_2,\ y_2) \\ &= (x_1 - x_2,\ y_1 - y_2)\,. \end{aligned} \tag{1-9}$$

■ **EXAMPLE 1.1** If $z_1 = (3,\ 7)$ and $z_2 = (5,\ -6)$, then

$$\begin{aligned} z_1 + z_2 &= (3,\ 7) + (5,\ -6) = (8,\ 1) \quad \textit{and} \\ z_1 - z_2 &= (3,\ 7) - (5,\ -6) = (-2,\ 13). \end{aligned}$$

We can also use the notation $z_1 = 3 + 7i$ and $z_2 = 5 - 6i$:

$$\begin{aligned} z_1 + z_2 &= (3 + 7i) + (5 - 6i) = 8 + i \quad \textit{and} \\ z_1 - z_2 &= (3 + 7i) - (5 - 6i) = -2 + 13i. \end{aligned}$$

Given the rationale we devised for addition and subtraction, it is tempting to define the product z_1z_2 as $z_1z_2 = (x_1x_2, y_1y_2)$. It turns out, however, that this is not a good definition, and we ask you in the exercises for this section to explain why. How, then, should products be defined? Again, if we equate (x, y) with $x + iy$ and assume, for the moment, that $i = \sqrt{-1}$ makes sense (so that $i^2 = -1$), we have

$$\begin{aligned} z_1z_2 &= (x_1, y_1)(x_2, y_2) \\ &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

Thus, it appears that we are forced into the following definition.

Definition 1.3: Multiplication

$$\begin{aligned} z_1z_2 &= (x_1, y_1)(x_2, y_2) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned} \quad (1\text{-}10)$$

■ **EXAMPLE 1.2** If $z_1 = (3, 7)$ and $z_2 = (5, -6)$, then

$$\begin{aligned} z_1z_2 &= (3, 7)(5, -6) \\ &= (3 \cdot 5 - 7(-6), 3(-6) + 5 \cdot 7) \\ &= (15 + 42, -18 + 35) \\ &= (57, 17). \end{aligned}$$

We get the same answer by using the notation $z_1 = 3 + 7i$ and $z_2 = 5 - 6i$:

$$\begin{aligned} z_1z_2 &= (3 + 7i)(5 - 6i) \\ &= 15 - 18i + 35i - 42i^2 \\ &= 15 - 42(-1) + (-18 + 35)i \\ &= 57 + 17i \\ &= (57, 17). \end{aligned}$$

Of course, it makes sense that the answer came out as we expected because we used the notation $x + iy$ as motivation for our definition in the first place.

To motivate our definition for division, we proceed along the same lines as we did for multiplication, assuming that $z_2 \neq 0$:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1,\ y_1)}{(x_2,\ y_2)}\\ &= \frac{(x_1 + iy_1)}{(x_2 + iy_2)}.\end{aligned}$$

We need to figure out a way to write the preceding quantity in the form $x + iy$. To do so, we use a standard trick and multiply the numerator and denominator by $x_2 - iy_2$, which gives

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}\\ &= \frac{x_1x_2 + y_1y_2 + i(-x_1y_2 + x_2y_1)}{x_2^2 + y_2^2}\\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2}\\ &= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2}\right).\end{aligned}$$

Thus, we finally arrive at a rather odd definition.

Definition 1.4: Division

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1, y_1)}{(x_2, y_2)}\\ &= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2}\right), \quad \text{for } z_2 \neq 0. \qquad (1\text{-}11)\end{aligned}$$

■ **EXAMPLE 1.3** If $z_1 = (3,\ 7)$ and $z_2 = (5,\ -6)$, then

$$\frac{z_1}{z_2} = \frac{(3, 7)}{(5, -6)} = \left(\frac{15 - 42}{25 + 36}, \frac{18 + 35}{25 + 36}\right) = \left(\frac{-27}{61}, \frac{53}{61}\right).$$

As with the example for multiplication, we also get this answer if we use the notation $x + iy$:

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{(3,7)}{(5,-6)} \\
&= \frac{3+7i}{5-6i} \\
&= \frac{3+7i}{5-6i}\,\frac{5+6i}{5+6i} \\
&= \frac{15+18i+35i+42i^2}{25+30i-30i-36i^2} \\
&= \frac{15-42+(18+35)\,i}{25+36} \\
&= \frac{-27}{61}+\frac{53}{61}i \\
&= \left(\frac{-27}{61},\frac{53}{61}\right).
\end{aligned}$$

To perform operations on complex numbers, most mathematicians would use the notation $x+iy$ and engage in algebraic manipulations, as we did here, rather than apply the complicated-looking definitions we gave for those operations on ordered pairs. This procedure is valid because we used the $x + iy$ notation as a guide for defining the operations in the first place. Remember, though, that the $x + iy$ notation is nothing more than a convenient bookkeeping device for keeping track of how to manipulate ordered pairs. It is the ordered pair algebraic definitions that form the real foundation on which the complex number system is based. In fact, if you were to program a computer to do arithmetic on complex numbers, your program would perform calculations on ordered pairs, using exactly the definitions that we gave.

Our algebraic definitions give complex numbers all the properties we normally ascribe to the real number system. Taken together, they describe what algebraists call a **field**. In formal terms, a field is a set (in this case, the complex numbers) together with two binary operations (in this case, addition and multiplication) having the following properties.

(P1) Commutative law for addition: $z_1 + z_2 = z_2 + z_1$.

(P2) Associative law for addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(P3) Additive identity: There is a complex number ω such that $z+\omega = z$ for all complex numbers z. The number ω is obviously the ordered pair $(0, 0)$.

(P4) Additive inverses: For any complex number z, there is a unique complex number η (depending on z) with the property that $z+\eta = (0,\ 0)$. Obviously, if $z = (x,\ y) = x + iy$, the number η will be $(-x,\ -y) = -x - iy = -z$.

(P5) Commutative law for multiplication: $z_1 z_2 = z_2 z_1$.

(P6) Associative law for multiplication: $z_1(z_2 z_3) = (z_1 z_2) z_3$.

(P7) Multiplicative identity: There is a complex number ζ such that $z\zeta = z$ for all complex numbers z. As you might expect, $(1,\ 0)$ is the unique complex number ζ having this property. We ask you to verify this identity in the exercises for this section.

(P8) Multiplicative inverses: For any complex number $z = (x, y)$ other than the number $(0,\ 0)$, there is a complex number (depending on z), which we denote z^{-1}, having the property that $zz^{-1} = (1,\ 0) = 1$. Based on our definition for division, it seems reasonable that the number z^{-1} would be $z^{-1} = \frac{(1,0)}{z} = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^y+y^2} + i\frac{-y}{x^2+y^2} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$. We ask you to confirm this result in the exercises for this section.

(P9) The distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

None of these properties is difficult to prove. Most of the proofs make use of corresponding facts in the real number system. To illustrate, we give a proof of property **(P1)**.

Proof of the commutative law for addition: Let $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$ be arbitrary complex numbers. Then,

$$\begin{aligned} z_1 + z_2 &= (x_1,\ y_1) + (x_2,\ y_2) \\ &= (x_1 + x_2,\ y_1 + y_2) && \text{(by definition of addition of complex numbers)} \\ &= (x_2 + x_1,\ y_2 + y_1) && \text{(by the commutative law for } \textit{real} \text{ numbers)} \\ &= (x_2,\ y_2) + (x_1,\ y_1) && \text{(by definition of addition of complex numbers)} \\ &= z_2 + z_1. \end{aligned}$$

Actually, you can think of the real number system as a subset of the complex number system. To see why, let's agree that, as any complex number of the form $(t,\ 0)$ is on the x-axis, we can identify it with the real number t. With this correspondence, we can easily verify that our definitions for addition, subtraction, multiplication, and division of complex numbers are consistent with the corresponding operations on real numbers. For example, if x_1 and x_2 are real numbers, then

$$\begin{aligned} x_1 x_2 &= (x_1,\ 0)(x_2,\ 0) && \text{(by our agreed correspondence)} \\ &= (x_1 x_2 - 0,\ 0 + 0) && \text{(by definition of multiplication of complex numbers)} \\ &= (x_1 x_2,\ 0) && \text{(confirming the consistence of our correspondence).} \end{aligned}$$

It is now time to show specifically how the symbol i relates to the quantity $\sqrt{-1}$. Note that

$$\begin{aligned}(0,\ 1)^2 &= (0,\ 1)(0,\ 1) \\ &= (0-1,\ 0+0) && \text{(by definition of multiplication of complex numbers)} \\ &= (-1,\ 0) \\ &= -1 && \text{(by our agreed correspondence).}\end{aligned}$$

If we use the symbol i for the point $(0,\ 1)$, the preceding identity gives

$$i^2 = (0,\ 1)^2 = -1,$$

which means $i = (0,\ 1) = \sqrt{-1}$. So, the next time you are having a discussion with your friends and they scoff when you claim that $\sqrt{-1}$ is not imaginary, calmly put your pencil on the point $(0,\ 1)$ of the coordinate plane and ask them if there is anything imaginary about it. When they agree there isn't, you can tell them that this point, in fact, represents the mysterious $\sqrt{-1}$ in the same way that $(1,\ 0)$ represents 1.

We can also see more clearly now how the notation $x+iy$ equates to $(x,\ y)$. Using the preceding conventions (i.e., $x = (x,\ 0)$, etc.), we have

$$\begin{aligned}x+iy &= (x,\ 0)+(0,\ 1)(y,\ 0) && \text{(by our previously discussed conventions)} \\ &= (x,\ 0)+(0,\ y) && \text{(by definition of multiplication of complex numbers)} \\ &= (x,\ y) && \text{(by definition of addition of complex numbers).}\end{aligned}$$

Thus, we may move freely between the notations $x+iy$ and $(x,\ y)$, depending on which is more convenient for the context in which we are working. Students sometimes wonder whether it matters where the "i" is located in writing a complex number. It does not. Generally, most texts place terms containing an "i" at the end of an expression, and place the "i" before a variable but after a constant. Thus, we write $x+iy$, $u+iv$, etc., but $3+7i, 5-6i$, and so forth. Because letters lower in the alphabet generally denote constants, you will usually (but not always) see the expression $a+bi$ instead of $a+ib$. Many authors write quantities like $1+i\sqrt{3}$ instead of $1+\sqrt{3}i$ to make sure the "i" is not mistakenly thought to be inside the square root symbol. Additionally, if there is concern that the "i" might be missed, it is sometimes placed before a lengthy expression, as in $2\cos\left(\frac{-5\pi}{6}+2n\pi\right)+i2\sin\left(\frac{-5\pi}{6}+2n\pi\right)$.

We close this section with three important definitions and a theorem involving them. We ask you for a proof of the theorem in the exercises.

Definition 1.5: Real part

The **real part of** z, denoted $\operatorname{Re}(z)$, is the real number x.

Definition 1.6: Imaginary part
The **imaginary part of** z, denoted $\operatorname{Im}(z)$, is the real number y.

Definition 1.7: Conjugate
The **conjugate of** z, denoted $\overline{z}$, is the complex number $(x, -y) = x - iy$.

■ **EXAMPLE 1.4** a) $\operatorname{Re}(-3+7i) = -3$ and $\operatorname{Re}[(9, 4)] = 9$. b) $\operatorname{Im}(-3+7i) = 7$ and $\operatorname{Im}[(9, 4)] = 4$. c) $\overline{-3+7i} = -3-7i$ and $\overline{(9, 4)} = (9, -4)$.

Theorem 1.1 *Suppose that z, z_1, and z_2 are arbitrary complex numbers. Then*

$$\overline{\overline{z}} = z. \tag{1-12}$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}. \tag{1-13}$$

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}. \tag{1-14}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \quad \text{if} \quad z_2 \neq 0. \tag{1-15}$$

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2}. \tag{1-16}$$

$$\operatorname{Im}(z) = \frac{z - \overline{z}}{2i}. \tag{1-17}$$

$$\operatorname{Re}(iz) = -\operatorname{Im}(z). \tag{1-18}$$

$$\operatorname{Im}(iz) = \operatorname{Re}(z). \tag{1-19}$$

Because of what it erroneously connotes, it is a shame that the term *imaginary* is used in Definition (1.6). It was coined by the brilliant mathematician and philosopher René Descartes (1596–1650) during an era when quantities such as $\sqrt{-1}$ were thought to be just that. Gauss, who was successful in getting mathematicians to adopt the phrase *complex number* rather than *imaginary number*, also suggested that they use *lateral part* of z in place of *imaginary part* of z. Unfortunately, that suggestion never caught on, and it appears we are stuck with what history has handed down to us.

EXERCISES FOR SECTION 1.2

1. Perform the required calculations and express your answers in the form $a + bi$.

(a) i^{275}.

(b) $\frac{1}{i^5}$.

(c) $\text{Re}\,(i)$.

(d) $\text{Im}(2)$.

(e) $(i-1)^3$.

(f) $(7-2i)(3i+5)$.

(g) $\text{Re}\,(7+6i) + \text{Im}\,(5-4i)$.

(h) $\text{Im}\left(\frac{1+2i}{3-4i}\right)$.

(i) $\frac{(4-i)(1-3i)}{-1+2i}$.

(j) $\overline{(1+i\sqrt{3})(i+\sqrt{3})}$.

2. Evaluate the following quantities.

(a) $\overline{(1+i)(2+i)}(3+i)$.

(b) $(3+i)/\overline{(2+i)}$.

(c) $\text{Re}\left[(i-1)^3\right]$.

(d) $\text{Im}[(1+i)^{-2}]$.

(e) $\frac{1+2i}{3-4i} - \frac{4-3i}{2-i}$.

(f) $(1+i)^{-2}$.

(g) $\text{Re}\left[(x-iy)^2\right]$.

(h) $\text{Im}\left(\frac{1}{x-iy}\right)$.

(i) $\text{Re}\left[(x+iy)\,(x-iy)\right]$.

(j) $\text{Im}\left[(x+iy)^3\right]$.

3. Show that $z\overline{z}$ is always a real number.

4. Verify Identities (1-12)–(1-19).

5. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree n.

(a) Suppose that $a_0, a_1, \ldots, a_{n-1}$ are all real. Show that if z_1 is a root of P, then $\overline{z_1}$ is also a root. In other words, the roots must be complex conjugates, something you likely learned without proof in high school.

(b) Suppose not all of $a_0, a_1, \ldots, a_{n-1}$ are real. Show that P has at least one root whose complex conjugate is not a root. *Hint*: Prove the contrapositive.

(c) Find an example of a polynomial that has some roots occurring as complex conjugates, and some not.

6. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be arbitrary complex numbers. Prove or disprove the following.

(a) $\operatorname{Re}(z_1 + z_2) = \operatorname{Re}(z_1) + \operatorname{Re}(z_2)$.

(b) $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2)$.

(c) $\operatorname{Im}(z_1 + z_2) = \operatorname{Im}(z_1) + \operatorname{Im}(z_2)$.

(d) $\operatorname{Im}(z_1 z_2) = \operatorname{Im}(z_1)\operatorname{Im}(z_2)$.

7. Prove that the complex number $(1, 0)$ (which we identify with the real number 1) is the multiplicative identity for complex numbers.

8. Use mathematical induction to show that the binomial theorem is valid for complex numbers. In other words, show that if z and w are arbitrary complex numbers and n is a positive integer, then $(z + w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

9. Let's use the symbol $*$ for a new type of multiplication of complex numbers defined by $z_1 * z_2 = (x_1 x_2, y_1 y_2)$. This exercise shows why this is an unfortunate definition.

(a) Use the definition given in property **(P7)** and state what the multiplicative identity ζ would have to be for this new multiplication.

(b) Show that if you use this new multiplication, nonzero complex numbers of the form $(0, a)$ have no inverse. That is, show that if $z = (0, a)$, there is no complex number w with the property that $z * w = \zeta$, where ζ is the multiplicative identity you found in part (a).

10. Explain why the complex number $(0, 0)$ (which, you recall, we identify with the real number 0) has no multiplicative inverse.

11. Prove property **(P9)**, the distributive law for complex numbers.

12. Verify that if $z = (x, y)$, with x and y not both 0, then $z^{-1} = \frac{(1,\,0)}{z}$ (i.e., $z^{-1} = \frac{1}{z}$). *Hint:* Let $z = (x, y)$ and use the (ordered pair) definition for division to compute $z^{-1} = \frac{(1,\,0)}{(x,\,y)}$. Then, with the result you obtained, use the (ordered pair) definition for multiplication to confirm that $zz^{-1} = (1, 0) = 1$.

13. From Exercise 12 and basic cancellation laws, it follows that $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}}$. The numerator here, $\overline{z}$, is trivial to calculate and, as the denominator $z\overline{z}$ is a real number (Exercise 3), computing the quotient $\frac{\overline{z}}{z\overline{z}}$ should be rather straightforward. Use this fact to compute z^{-1} if $z = 2 + 3i$ and again if $z = 7 - 5i$.