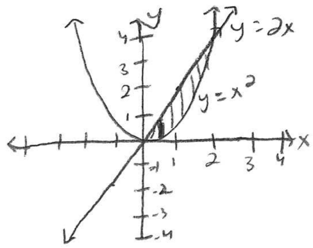
Exam 1 Calc 2 1/29/2016

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an integral for the area of the region bounded between $y = 2x$ and $y = x^2$.



Area $= \int_0^2 (2x - x^2)\,dx$

$2x = x^2$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\left.\begin{array}{l} x = 0 \\ x = 2 \end{array}\right\}$ pt of intersection

Great!

2. Evaluate $\int \frac{x}{\sqrt{x^2+9}}\,dx$.

$u = x^2 + 9$

$\frac{du}{dx} = 2x$

$\frac{1}{2x}\,du = dx$

$\int \frac{x}{\sqrt{u}} \frac{1}{2x}\,du$

$\frac{1}{2} \int \frac{1}{\sqrt{u}}\,du$

$\frac{1}{2} \int u^{-\frac{1}{2}}\,du$

$\frac{1}{2}\left[2u^{\frac{1}{2}}\right]$

$u^{\frac{1}{2}} + C$

$\sqrt{x^2+9} + C$

Great

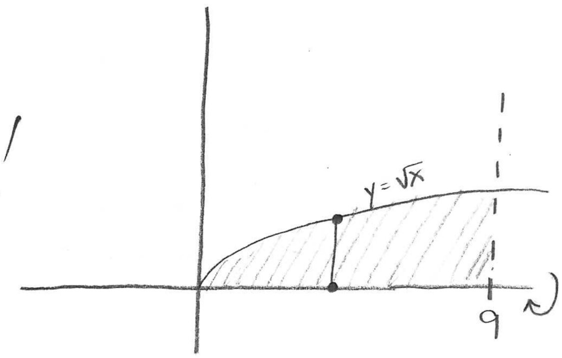
3. Suppose the region between $y = \sqrt{x}$ and the x-axis on the interval from $x = 0$ to $x = 9$ is rotated around the x-axis. Set up an integral for the volume of the solid produced.

$$\text{Volume} = \pi \int_a^b [F(x)]^2 \, dx$$

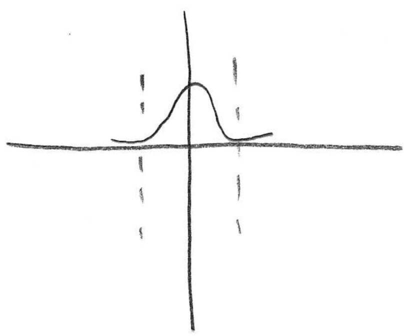
$$= \pi \int_0^9 (\sqrt{x})^2 \, dx$$

$$\boxed{= \pi \int_0^9 x \, dx}$$

Excellent!



4. Set up an integral for the average value of the "bump" function $y = \dfrac{1}{4+x^2}$ on the interval $[-2, 2]$.



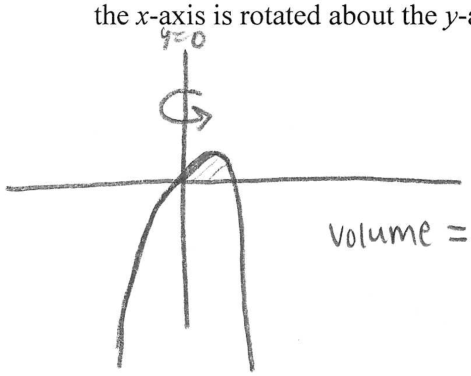
$$\text{Average value} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

$$\frac{1}{2-(-2)} \int_{-2}^{2} \left(\frac{1}{4+x^2} \right) dy$$

Excellent!

$$\text{Average value} = \frac{1}{4} \int_{-2}^{2} \left(\frac{1}{4+x^2} \right) dx$$

5. Write an integral for the volume obtained when the region bounded between $y = x - x^{12}$ and the x-axis is rotated about the y-axis.



Shell!

Volume = $\boxed{2\pi \int_0^1 (x)(x - x^{12})\ dx}$

radius height width

Great!

$x - x^{12} = 0$

$x(1 - x^{12})$

$1 = x^{12}$

$\sqrt[12]{1} = \sqrt[12]{x^{12}}$

6. Evaluate $\displaystyle\int \frac{(\ln x)^4}{x}\,dx$.

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$= \int \frac{u^4}{x}\,dx$

$= \int u^4 \cdot \frac{1}{x} \cdot dx$

$= \int u^4 \cdot \frac{dx}{x}$

$= \int u^4\,du$

$= \left(\frac{u^5}{5}\right) + C$

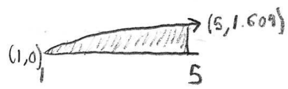
$\boxed{= \frac{(\ln x)^5}{5} + C}$

Excellent

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Whoa! Calc 1 was okay, but now in Calc 2 there's all these guys who just know a lot more stuff than me and it's not even fair. On our first exam I kinda failed, and after it there were these two guys who sit in front of me joking about how easy it was 'cause it was multiple choice and they knew a bunch of the answers couldn't work so they knew the right answer without even working it out. I'm really pissed 'cause I spent like ten minutes on that one and still couldn't get it. So how the heck was I supposed to know without working it all out that the integral of ln x from 1 to 5 is less than 7 so all the other choices are wrong?

Help Biff out. Explain to him as clearly as possible how other students might have drawn their conclusion.

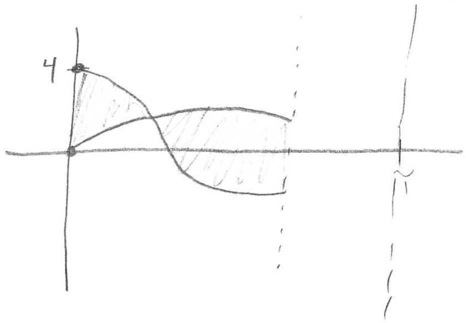
$\int_1^5 \ln x \, dx$



The integral can be thought of as the sum of its parts, or in other words, the area below the curve. Using simple estimation knowing what the graph of $y = \ln(x)$ looks like, one can figure the integral is less than 7. Taking the max value of $y = \ln(x)$ [1, 5] and multiplying it by the range $5-1=4$, one finds the product to be about 6.4, which is less than 7. Furtermore it's a large overestimate. This could be used to help answer a multiple choice question.

Exactly.

8. Take the region enclosed between $y = 2\sin(x)$ and $y = 4\cos(x)$ from $x = 0$ to $x = 0.7\pi$, and rotate it around the line $x = \pi$. Write an integral for the volume of the resulting solid.



$2\sin x = 4\cos x$

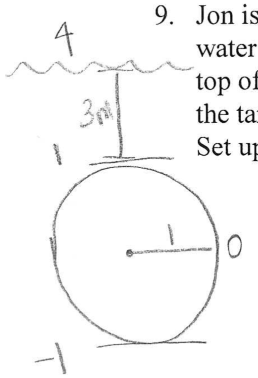
$\tan x = 2$

$x = ac$

$$\left[2\pi \int_0^{\arctan(2)} \left[4\cos(x) - 2\sin(x)\right]\left[\pi - x\right] dx\right] + \left[2\pi \int_{\arctan(2)}^{.7\pi} \left[2\sin(x) - 4\cos(x)\right]\left[\pi - x\right] dx\right]$$

Well done.

9. Jon is planning for the apocalypse by burying a spherical tank in his yard as an emergency water supply. The tank will be 1m in radius, with a tube leading up to a level 3m above the top of the tank. Jon needs to know how much work is required to pump all of the water in the tank out through the tube, so he can determine the appropriate battery to power the pump. Set up an integral for this amount of work.



$r = \sqrt{1-x^2}$ m

$A = \pi(1-x^2)$ m^2

$V = \pi(1-x^2)\Delta x$ m^3

$m = \pi(1-x^2)\Delta x \; 1{,}000$ m^{3}Kg

$F = \pi(1-x^2)\Delta x \; 1{,}000 \cdot 9.8$ N

$W = \pi(1-x^2)\Delta x \; 1{,}000 \cdot 9.8 (4-x)$ J

Nice Work!

$\boxed{\text{Total } W \rightarrow \pi \int_{-1}^{1} (1-x^2)(4-x)(9800)\,dx \text{ J}}$

10. Let $a > 0$. Show that the volume obtained when the region between $y = a\sqrt{x - ax^2}$ and the x-axis is rotated around the x-axis is independent of the constant a.

Intersects axis where

$$a\sqrt{x - ax^2} = 0$$

$$a^2(x - ax^2) = 0^2$$

$$x - ax^2 = 0$$

$$x(1 - ax) = 0$$

$$x = 0 \text{ or } x = 1/a$$

$$\text{Volume} = \pi \int_0^{1/a} \left(a\sqrt{x - ax^2}\right)^2 dx$$

$$= \pi \int_0^{1/a} a^2(x - ax^2)\, dx$$

$$= \pi \int_0^{1/a} (a^2 x - a^3 x^2)\, dx$$

$$= \pi \left[a^2 \cdot \frac{x^2}{2} - a^3 \cdot \frac{x^3}{3} \right]_0^{1/a}$$

$$= \pi \left(a^2 \cdot \frac{1}{2a^2} - a^3 \cdot \frac{1}{3a^3} \right)$$

$$= \pi \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \pi \cdot \frac{1}{6}$$

$$= \frac{\pi}{6}$$

So we get the same volume no matter what a is!