Examlet 3 Foundations of Advanced Math 3/25/16

1. a) State the definition of an injection.

$f: A \to B$ is injective iff $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

Good

b) State the definition of a surjection.

$f: A \to B$ is surjective iff $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

Great

c) State the definition of equipollent sets.

Set A and set B are equipollent iff there exists bijection $f: A \to B$.

Great

d) State the definition of a denumerable set.

Set A is denumerable iff A and $\mathbb{N}$ are equipollent.

Great

e) State the definition of a countable set.

Set A is countable iff A and some subset of $\mathbb{N}$ are equipollent.

Great

2. a) If $f: A \to B$ has an inverse function g, then g has f as an inverse function also.

$f: A \to B$ has inverse function g, so $g: B \to A$, $\forall a \in A$, $g(f(a)) = a$ and $f(g(b)) = b$ $\forall b \in B$. Let g have inverse function h, so $h: A \to B$, $\forall a \in A$, $g(h(a)) = a$ and $\forall b \in B$, $h(g(b)) = b$. f has all the same qualities as h, so f is the inverse of g when $f: A \to B$ has inverse function g. $\square$

Good

b) Give an example of functions $f: A \to B$ and $g: B \to A$ where $\exists a \in A$ such that $g \circ f(a) = a$, but g is not an inverse function for f.

Let $f: \overset{(A)}{\mathbb{R}} \to \overset{(B)}{\mathbb{R}}$

$$\boxed{\begin{array}{l} A = \mathbb{R} \\ B = \mathbb{R} \end{array}}$$

$f(x) = x$

Let $g: \overset{(B)}{\mathbb{R}} \to \overset{(A)}{\mathbb{R}}$

$g(x) = x^3$

let $a = 1$

$a \in A$ $(a \in \mathbb{R})$

$g \circ f(a) = g \circ f(1) = g(1) = 1 = a$, so $\exists a \in A$ $g \circ f(a) = a$

$g \circ f(a) = a$, yet x^3 is not the inverse of x, and g is not the inverse of f. $\square$

Great

3. If $f:A \to B$ and $g:B \to C$ are injective functions, then $g \circ f$ is injective.

Well, Suppose $g \circ f(a_1) = g \circ f(a_2)$.

By composition we can write this as $g(f(a_1)) = g(f(a_2))$.

Because g is injective, $f(a_1) = f(a_2)$.

Because f is injective, $a_1 = a_2$.

Therefore, $g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$.

This means that $g \circ f$ is injective. $\square$

Great

4. The set of integers, $\mathbb{Z}$, is denumerable.

$$f(x) = \begin{cases} 2|x| & x \leq 0 \\ 2x-1 & x > 0 \end{cases}$$

is a bijective function where all the positive integers are sent to the ~~even~~ naturals and all the negative integers get sent to the odd integers, and zero gets sent to zero

so because a bijection exists between $\mathbb{Z}$ and $\mathbb{N}$, $\mathbb{Z}$ is denumerable

Great.

5. If A is equipollent to B, and B is equipollent to C, then A is equipollent to C.

If A is equipollent to B then there exists a bijection $f: A \to B$
If B is equipollent to C then there exists a bijection $g: B \to C$
By previous proofs we know that the composition of two bijections is a bijection, so $g \circ f: A \to C$ is a bijection. Since there exists a bijection from A to C we know that A is equipollent to C. $\square$

Excellent