**Exam 1** **Foundations of Advanced Math** **2/3/17**

1. The sum of two odd integers is even.

proooof: supose you have two odd integers $x$ and $y \in \mathbb{Z}$

this means $x = 2n+1$ for some $n \in \mathbb{Z}$

and $y = 2m+1$ for some $m \in \mathbb{Z}$

$x+y = 2n+1+2m+1$

$x+y = 2n+2m+2$

$x+y = 2(n+m+1)$

$n, m, 1 \in \mathbb{Z}$ so $n+m+1$ is an int by closure

$2(n+m+1)$ is therefore an even number by def

So $x+y$ or the sum of two odd integers is even. $\square$

Excellent!

2. If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

Proof:

Given $a \equiv_m b$ and $b \equiv_m c$, we know that $m | (b-a)$ and $m | (c-b)$. So $b-a$ can be written as $b-a = pm$ with $p \in \mathbb{Z}$, and $c-b$ can be written as $c-b = qm$ with $q \in \mathbb{Z}$. If we solve for $b$ in the second equation and substitute that into the first, we get $c-qm-a = pm$, and adding $qm$ to both sides results in $c-a = pm+qm = m(p+q)$. Since $p+q \in \mathbb{Z}$ due to closure under addition, this means $m | (c-a)$, which proves $a \equiv_m c$ by definition. $\square$

Great.

$(n+1)(2n+1) = 2n^2+n+2n+1 = (2n^2+3n+1)n = 2n^3+3n^2+n$

3. For any $n \in \mathbb{Z}, 6|n(n+1)(2n+1)$

Use Induction

Base case: $n=0$ so $6 \overset{?}{|} 0(0+1)(2(0)+1)$, $6|0$ is true.

Suppose it is true for $k$ such that $6|k(k+1)(2k+1)$.

so we can say $2k^3+3k^2+k = 6 \cdot x$ for $x \in \mathbb{Z}$.

So we have to prove $6|(k+1)(k+1+1)(2(k+1)+1)$

simplified $6|(k+1)(k+2)(2k+3)$ → scratch work

$(k+2)(2k+3)$

$2k^2+3k+4k+6$

$(k+1)(2k^2+7k+6)$

$\cancel{2k^3}+\cancel{7k^2}+6k+2k^2+\cancel{7k}+6$

$= 2k^3+9k^2+13k+6$

this can be rewritten $2k^3+9k^2+13k+6$

breaking down left side results in $(2k^3+3k^2+k)+6k^2+12k+6$

through substitution we get $6x+6k^2+12k+6$

factored $6(x+k^2+2k+1)$

Since $x+k^2+2k+1$ is an integer by closure, (call it $a$, $a \in \mathbb{Z}$)
$6a$ is divisible by six, so $6|(k+1)((k+1)+1)(2(k+1)+1)$.

Since it is true for our base case and when we suppose its true that $n=k$ it is alway true for $n=k+1$ we can say by mathematical induction that $6|n(n+1)(2n+1)$. □

Nice.

4. Determine whether the statements $P \Rightarrow Q$ and $\neg P \vee Q$ are logically equivalent.

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg P$ | $\neg P \vee Q$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | T | F | T | T |

Since the truth values for the starred columns are the same for all inputs, we can say the statements $P \Rightarrow Q$ and $\neg P \vee Q$ are logically equivalent. $\square$

Good

5. $\sqrt{3}$ is irrational.

Suppose that $\sqrt{3}$ is rational

Then $\sqrt{3} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$, $q \neq 0$

↳ if necessary, reduce the fraction so p and q have no common factors

$\frac{p^2}{q^2} = 3$

$p^2 = 3q^2$

$q^2$ is an integer by closure, so $p^2$ is three times an integer

By definition, $p^2$ is threven

Since $p^2$ is threven, p must be threven as well (by previous results*)

So $p = 3m$ for some $m \in \mathbb{Z}$

$p^2 = 3q^2$

$(3m)^2 = 3q^2$

$9m^2 = 3q^2$

$q^2 = 3m^2$

$m^2$ is an integer by closure, so $q^2$ is three times an integer.

By definition, $q^2$ is threven.

Since $q^2$ is threven, q must be threven as well (by previous results)

Since both p and q are threven, the share a common factor of 3. However, we stated earlier that p and q have no common factors, so there is a contradiction. That's why our assumption that $\sqrt{3}$ is rational is wrong. Therefore, $\sqrt{3}$ is irrational. □

* We previously proved that square of threven is threven, square of throdd is throdd, and square of throddodd is throdd. Since every integer is either threven, throdd, or throddodd, it is just a threven integer that will give us threven if we square it.

Well done!