Examlet 3b Foundations of Advanced Math 3/24/17

1. a) State the definition of sets A and B being **equipollent**.

We say A is equipollent to B if there exists a bijection from A to B. $f: A \to B$ - which is bijective.

b) Give five distinct examples of sets equipollent to $\mathbb{N}$.

1. negative $\mathbb{Z}$
2. positive $\mathbb{Z}$
3. odd numbers
4. even numbers
5. $\mathbb{N}$

Good

2. The composition of two injective functions is injective.

Let $f: A \to B$ be injective so

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Let $g: B \to C$ be injective so

$g(a_1) = g(a_2) \Rightarrow a_1 = a_2$

Suppose $g \circ f(a_1) = g \circ f(a_2)$

Rewritten: $g(f(a_1)) = g(f(a_2))$

Because g is injective, $g(f(a_1)) = g(f(a_2)) \Rightarrow f(a_1) = f(a_2)$

Because f is injective, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

So $g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$

So by definition, the composition of two injective functions is injective. $\square$

Excellent!

3. The composition of two surjective functions is surjective.

Let $f: A \to B$ be surjective, so
$\forall\, b \in B$, $\exists\, a \in A$ such that $f(a) = b$

Let $g: B \to C$ be surjective, so
$\forall\, c \in C$, $\exists\, b \in B$ such that $g(b) = c$

So we know $\forall c \in C$, $\exists\, b \in B$ such that $g(b) = c$
Taking that some b,
we know for that $b \in B$, $\exists\, a \in A$ such that $f(a) = b$

So $g(f(a)) = c$ $\quad g(b) = c$

So. $\forall\, c \in C$, $\exists\, a \in A$ such that $g(f(a)) = c$

So by definition, the composition of two surjective functions is surjective. $\square$

Excellent!

4. Let $f(x) = \sqrt{9x+5}$. What is the inverse function for f, and what are its domain and codomain?

The inverse of $f(x) = \sqrt{9x+5}$

is $f^{-1}(x) = \frac{x^2-5}{9}$.

Well, since the domain for $f(x)$ is $[-\frac{9}{5}, \infty)$,

& the codomain for $f(x)$ is $[0, \infty)$;

the domain for $f^{-1}(2)$ is $[0, \infty)$ &

the codomain is $[-9/5, \infty)$.

Excellent!

Scratch work

$-\frac{9}{5}(a)+5 = \ldots$

$g = \frac{y^2-5}{9}$

$f(4) = \sqrt{9(4)+5}$

$g = \frac{(\sqrt{9(4)+5})^2 - 5}{9}$

$= \frac{9(4)+5-5}{9} = 4$

f_2 -

$f_1(-x) = -f_1(x)$

5. a) Let $f_1: \mathbb{R} \to \mathbb{R}$ and $f_2: \mathbb{R} \to \mathbb{R}$ be odd functions. Then $f_1 + f_2$ is an odd function.

Since f_1 is odd we know $f_1(-x) = -f_1(x)$ and since f_2 is odd we know $f_2(-x) = -f_2(x)$

So $f_1(-x) + f_2(-x) = -f_1(x) + -f_2(x)$

We can then undistribute the -1 and get $-(f_1(x) + f_2(x))$

We also know $f_1(-x) + f_2(-x)$ can be written as $(f_1 + f_2)(-x)$

So, $(f_1 + f_2)(-x) = -((f_1 + f_2)(x))$ so $f_1 + f_2$ is odd by definition

Good

b) Let $n \in \mathbb{N}$, and $f_i: \mathbb{R} \to \mathbb{R}$ be an odd function for each $i \in \{x \in \mathbb{N} \mid 1 \le x \le n\}$. Then $\sum_{i=1}^{n} f_i$ is an odd function.

Induction! $n = 1$ $\sum_{i=1}^{1} f_i = f_i$ which is an odd function. So we can assume $\sum_{i=1}^{k} f_i$ is odd. Now let's test $\sum_{i=1}^{k+1} f_i$

Another way this can be written is $\sum_{i=1}^{k} f_i + f_{k+1}$

We know $\sum_{i=1}^{k} f_i$ is odd because of our inductive hypothesis.

We also know f_{k+1} is odd. Since we just proved in step a the addition of two odd functions is odd. $\sum_{i=1}^{n} f_i$ must be odd.

Nice!