Examlet 2　　Advanced Geometry　　3/1/17

1. a) State the definition of a *scalene triangle*.

A triangle is scalene iff its three edges all have different lengths.

b) State the definition of a *quadrilateral*.

Let A, B, C, and D be four distinct points, no three of which are collinear, and that no two of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have points in common except endpoints. Then the union of these four segments is a quadrilateral.

c) State the Saccheri-Legendre Theorem.

If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) \leq 180°$

d) State the Scalene Inequality.

In any triangle, the greater side is opposite the greater angle and the greater angle lies opposite the greater side.

e) State the Universal Hyperbolic Theorem.

If there exists a line l_0 with an external point P_0 and at least two lines through P_0 parallel to l_0, then for every line l and every external point P there are at least two lines through P parallel to l.

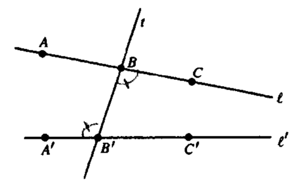
2. Which of the following are equivalent (given the other postulates of neutral geometry) to the Euclidean Parallel Postulate? Check all that apply.

- ☐ The double perpendicular construction
- ☐ The Saccheri-Legendre Theorem
- ☒ Existence of rectangles
- ☒ Euclid's Postulate V
- ☒ Converse of the Alternate Interior Angles Theorem
- ☒ If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180°$.
- ☒ Clairaut's Axiom
- ☐ A unicorn ate my petunias.
- ☒ There exists a triangle whose defect is 0°.
- ☐ The Universal Hyperbolic Theorem

3. Provide good justifications in the blanks below for the corresponding statements:

Alternate Interior Angle Theorem.

Proposition: If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'.



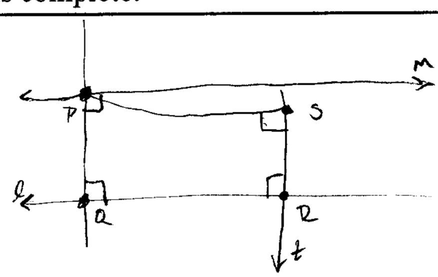
Statement:	Reason:
Let ℓ and ℓ' be two lines cut by transversal t such that a pair of alternate interior angles is congruent.	Hypothesis
Choose points A, B, C, and A', B', C' as in the figure above. Suppose $\angle A'B'B \cong \angle B'BC$.	Ruler/Point Construction Post. Hypothesis
We must prove that ℓ is parallel to ℓ'. Suppose there exists a point D such that D lies on both ℓ and ℓ'.	RAA Hypothesis
If D lies on the same side of t as C, then $\angle A'B'B$ is an exterior angle for $\triangle BB'D$,	Definition of Exterior Angle.
while $\angle B'BC$ is a remote interior angle for $\triangle BB'D$.	Definition of Remote Interior Angle.
This is a contradiction.	Exterior Angle Theorem.
In case D lies on the same side of t as A, then $\angle B'BC$ is an exterior angle and $\angle A'B'B$ is a remote interior angle for $\triangle BB'D$,	Definitions of Exterior Angle and Remote Interior Angle.
and again we have a contradiction.	Exterior Angle Theorem
Since D must lie on one of the two sides of t,	Plane Separation Postulate.
we are forced to conclude that the proposition holds.	Contradiction & Definition of Parallel Lines.

Great

4. Provide good justifications in the blanks below for the corresponding statements:

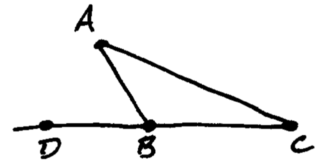
Proposition: If there exists one line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0, then for every line ℓ and for every external point P there exist at least two lines that pass through P and are parallel to ℓ.

Statement:	Reason:
S'pose there exists a line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0.	Hypothesis
Then the Euclidean Parallel Postulate fails.	Euclidean Parallel Postulate
No rectangle exists.	Clairaut's Axiom is equivalent to Euclidean Parallel Postulate
Let ℓ be a line and P an external point.	Hypothesis
We must prove that there are at least two lines through P that are both parallel to ℓ. Drop a perpendicular to ℓ through P and call the foot of that perpendicular Q.	Existence & Uniqueness of Perpendiculars
Let m be the line through P that is perpendicular to $\overleftrightarrow{PQ}$.	Existence & Uniqueness of Perpendiculars
Choose a point R on ℓ that is different from Q and let t be the line through R that is perpendicular to ℓ.	Existence & Uniqueness of Perpendiculars
Drop a perpendicular from P to t and call the foot of the perpendicular S.	Existence & Uniqueness of Perpendiculars
Now $\square PQRS$ is a Lambert quadrilateral.	Definition of Lambert Quadrilateral
But $\square PQRS$ is not a rectangle (reason?), so $\angle QPS$ is not a right angle and $\overleftrightarrow{PS} \neq m$.	Since shown no rectangles exist earlier in proof.
Nevertheless $\overleftrightarrow{PS}$ is parallel to ℓ,	Alternate Interior Angles
so our proof is complete.	Because our proof is complete.



Excellent

5. a) Prove or give a counterexample: If one interior angle of a triangle is obtuse, then both the other interior angles are acute.



Let $\triangle ABC$ be a triangle with $\angle ABC$ obtuse.
Take a point D on $\overleftrightarrow{CB}$ for which $D*B*C$.
Then $\angle ABD$ forms a linear pair with $\angle ABC$,
so $\mu\angle ABC + \mu\angle ABD = 180°$, and since $\mu\angle ABC > 90°$ by def. obtuse,
we must have $\mu\angle ABD < 90°$. But $\angle ABD$ is an external angle for
$\triangle ABC$, so the remote interior angles $\angle BAC$ and $\angle BCA$ are both
less by the exterior angles theorem, so $90° > \mu\angle ABD > \mu\angle BAC$
$\Rightarrow 90° > \mu\angle BAC$ and $90° > \mu\angle ABD > \mu\angle BCA \Rightarrow 90° > \mu\angle BCA$,
as desired.

b) Prove or give a counterexample: If one interior angle of a triangle is acute, then at least one of the other interior angles is obtuse.

False. Consider an equilateral triangle in the Euclidean plane, so all three $60°$ angles are acute.