Examlet 3 **Advanced Geometry** **4/7/17**

1. a) State the Neutral Area Postulate.

Associated with each polygonal region R is a nonnegative number $\alpha(R)$ called the area of R, such that it satisfies the two following criteria:

1) Congruence - If two triangles are congruent, then their associated triangular regions are also congruent

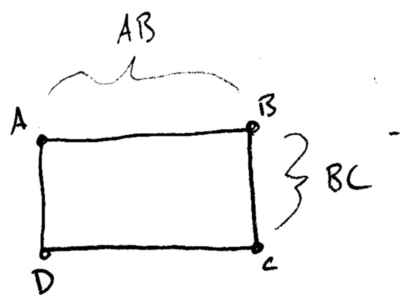
2) Additivity - If the region R is composed of two non overlapping regions R_1 + R_2 then $\alpha(R) = \alpha(R_1) + \alpha(R_2)$

b) State the Euclidean Area Postulate.

Great

if $\square ABCD$ is a rectangle, then

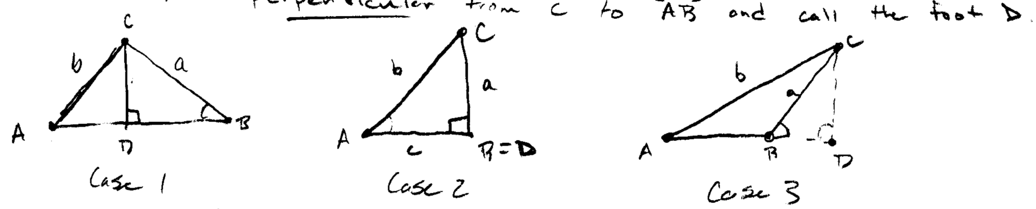
$$\alpha(\square ABCD) = AB \cdot BC$$



2. State and prove the Law of Sines.

For any triangle $\triangle ABC$, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Proof. Let $\triangle ABC$ be any triangle with a vertex at B. For any triangle there is at most one nonacute angle. Let $\angle A$ be an acute angle. Thus, $\angle B$ is either acute, right or obtuse. Drop a perpendicular from C to $\overleftrightarrow{AB}$ and call the foot D.



Case 1: $\angle B$ is acute (Add). By definition, $\sin B = \frac{CD}{a}$ and $\sin A = \frac{CD}{b}$. So $CD = a \sin B = b \sin A$. Thus, $\frac{\sin A}{a} = \frac{\sin B}{b}$. (Add) By Lemma 48.6, D is in the interior of $\overline{AB}$.

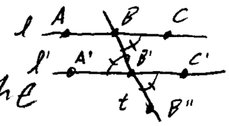
Case 2: $\angle B$ is right. By definition of sine, we have $\sin A = \frac{CD}{b}$ and $\sin B = 1$ since B is a right angle. So $CD = b \sin A$ and we have $CD = CB = a$, thus $1 \cdot a = b \sin A$. Then $\sin B \cdot a = b \sin A$ and we get $\frac{\sin A}{a} = \frac{\sin B}{b}$.

Case 3: $\angle B$ is obtuse. By definition of sine for obtuse angle we get $\sin\theta = \sin\theta'$. Hence $\sin B = \frac{CD}{a}$ and using right triangle $\triangle ACD$ we get $\sin A = \frac{CD}{b}$. Thus $CD = a \sin B = b \sin A$ and by Algebra we get $\frac{\sin B}{b} = \frac{\sin A}{a}$.

To show $\frac{\sin B}{b} = \frac{\sin C}{c}$, we just switch vertices A and C for the proof above. Thus $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and proof is complete. $\square$

Great

3. Show that in the Euclidean plane if ℓ and ℓ' are lines cut by a transversal t and ℓ is parallel to ℓ', then two corresponding angles are congruent.



Given two parallel lines l and l' in the Euclidean plane, cut by a transversal t (hypothesis) we will now label points for ease (Ruler post). Let A, B, C lie on l such that $A * B * C$ and B is the intersection of t and l. Let A', B', C' lie on l' such that $A' * B' * C'$ and B' is the intersection of t and l'. Let A and A' lie on the same side of t. Now label a point B'' on t such that

$B * B' * B''$. By the converse to Alternate Interior angles theorem, we know $\mu(\angle A'B'B) = \mu(\angle B'BC)$. By Vertical angles theorem, we know $\angle A'B'B \cong \angle B''B'C'$ hence by transitivity $\angle B'BC \cong \angle B''B'C'$ which are by definition corresponding angles. The other angles would follow similarly, hence our proof is complete.

Great

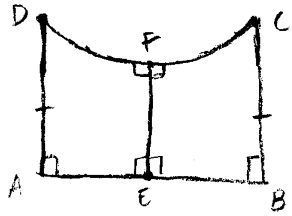
4. State and prove the Pythagorean Theorem (using similar triangles).

If $\triangle ABC$ is a right triangle with the right angle at vertex C, then $a^2+b^2=c^2$.

Proof:
- Let $\triangle ABC$ be a triangle with the right angle at vertex C.
- Drop a perpendicular from C to $\overleftrightarrow{AB}$ and call the foot D.
- By lemma 4.8.6, D is in the interior of $\overline{AB}$.
- Now we know $\mu(\angle A)+\mu(\angle B)=90^\circ$ and $\mu(\angle A)+\mu(\angle ACD)=90^\circ$ (Angle Sum Thm)
- So $\angle B \cong \angle ACD$ and $\angle A \cong \angle BCD$ (Algebra)
- Hence, we have three similar triangles: $\triangle ABC \sim \triangle ACD \sim \triangle BCD$.
- Let $x=AD$, $y=BD$ and $h=CD$.
- Applying the Similar Triangle Theorem, we get $\frac{x}{b}=\frac{b}{c}$ and $\frac{y}{a}=\frac{a}{c}$
- So $b^2=xc$ and $a^2=yc$
- Adding them together we get $a^2+b^2=c(x+y)$
- Since $c=x+y$, we have $a^2+b^2=c^2$ and the proof is complete. $\square$

Nice!

5. Prove that in the hyperbolic plane, a Saccheri quadrilateral must have the length of its altitude less than the length of its side.



Proof: Let $\square ABCD$ be a Saccheri quadrilateral

Drop a perpendicular line from the midpoint of $\overleftrightarrow{CD}$ to the midpoint of $\overleftrightarrow{AB}$. (existence of midpoint and perpendicular)

Call the midpoints of $\overleftrightarrow{AB}$ E and $\overleftrightarrow{CD}$ F.

Both $\square AEFD$ and $\square BCFE$ are Lambert quadrilaterals (def. of Lambert quadrilateral)

From Lambert quadrilaterals we know the length of the side with two right angles is less than the side without, otherwise it would be a rectangle, but rectangles don't exist in hyperbolic geometry.

$\therefore EF < BC$ and $EF < AD$

Excellent