Exam 1 **Advanced Geometry** **1/31/18**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Evaluate

$$\int \frac{2+x^2}{1+x^2}\,dx$$

$$= \int \left(\frac{1+x^2}{1+x^2} + \frac{1}{1+x^2} \right) dx$$

$$= \int \left(1 + \frac{1}{1+x^2} \right) dx$$

$$= x + \arctan x + C$$

Great

2. Evaluate

$$\int x e^x\,dx$$

$$\int x e^x\,dx$$

$u = x$ $\quad v = e^x$

$u' = 1\,dx$ $\quad v' = e^x\,dx$

$$= xe^x - \int e^x\,dx$$

$$= \boxed{xe^x - e^x + C}$$

Good

3. Evaluate

$$\int x^2 \sqrt{x^3 - 8}\, dx$$

$= \int x^2 \sqrt{u} \; \frac{du}{3x^2}$

$= \int \cancel{x^2} \cdot u^{1/2} \cdot \frac{du}{3\cancel{x^2}}$

$= \frac{1}{3}\left(\frac{2u^{3/2}}{3} \right) + C$

$= \boxed{\frac{2(x^3-8)^{3/2}}{9} + C}$

$u = x^3 - 8$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

Great

4. Evaluate

$$\int_3^\infty \frac{1}{(x-2)^{3/2}}\, dx$$

$\lim_{b \to \infty} \int_3^b \frac{1}{(x-2)^{3/2}} dx = \lim_{b \to \infty} \int_3^b (x-2)^{-3/2} dx$

$\lim_{b \to \infty} \frac{(x-2)^{-\frac{3}{2} + \frac{2}{2}}}{-\frac{3}{2} + \frac{2}{2}} = \lim_{b \to \infty} -2(x-2)^{-\frac{1}{2}} \Big|_3^b$

Excellent!

$= \lim_{b \to \infty} \frac{-2}{\sqrt{b-2}} - \frac{-2}{\sqrt{3-2}} = 0 + \frac{+2}{\sqrt{1}}$

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5. Evaluate

$$\int \sec^4\theta \tan\theta \, d\theta$$

$U = \tan\theta$
$dU = \sec^2\theta \, d\theta$

$\int \sec^2\theta(\tan^2\theta+1)\tan\theta \, d\theta \rightarrow \int \frac{\cancel{\sec^2\theta}(U^2+1)U\,dU}{\cancel{\sec^2\theta}} \rightarrow \int U^3 + U \, dU$

$\int U^3 dU + \int U \, dU \rightarrow \frac{1}{4}U^4 + C \quad + \frac{1}{2}U^2 + C$

$\boxed{\frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta + 2C}$

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$\frac{d}{dx}\left(\frac{1}{4}(\tan\theta)^4 + \frac{1}{2}(\tan\theta)^2 + 2C\right)$

$= \tan^3\theta\sec^2\theta + \tan\theta\sec^2\theta$

$= \sec^2\theta(\tan\theta[(\sec^2\theta-1)+1])$

$= \sec^2\theta(\sec^2\theta\tan\theta) \rightarrow \sec^4\theta\tan\theta$

Excellent!

6. Evaluate

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx$$

$(x+1)(2x+1)$

I wish:

$\frac{2}{2x^2+3x+1}(2x^2+3x+1) = \frac{A}{x+1}(2x^2+3x+1) + \frac{B}{2x+1}(2x^2+3x+1)$

$2 = 2Ax + A + Bx + B$

$2 = A + B$
$2 - B = A$

$2-(4) = A$
$-2 = A$

$0x = 2Ax + Bx$
$0x = 2(2-B)x + Bx$
$0x = 4x - 2Bx + Bx$
$0x = 4x - Bx$
$0 = 4x - Bx$
$Bx = 4x$
$B = 4$

$\int_0^1 \frac{-2}{x+1} + \int_0^1 \frac{4}{2x+1}$

$-2\int_0^1 \frac{1}{x+1} + 4\int_0^1 \frac{1}{2x+1}$

$U = 2x+1$
$dU = 2dx$

$-2[\ln|(1)+1| - \ln|(0)+1|] + \frac{4}{2}\int_{x=0}^{x=1} \frac{1}{U}\,dU$

$-2[\ln|2| - 0] + 2[\ln|2(1)+1| - \ln|2(0)+1|]$

$-2\ln 2 + 2[\ln 3 - 0]$

$\boxed{-2\ln 2 + 2\ln 3}$

Very nice!

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, Calc is tough! I thought I had it all figured out, but I guess it's just too much for me. We had this assignment and I, like, outsourced it to Mathematica, right? So for this one where we were supposed to integrate 1 over $3x-2$, Mathematica said $\frac{1}{3}\ln(3x-2)$, so I wrote that down. But the grader took off points and wrote this nasty note about something general and some domain thing, and about how even a computer could do as well as I did, like that was a bad thing. But dude, I think computers are automatically right, right?"

Help Biff out by explaining what shortcomings there might be to his answer, and how he should improve it.

$$\int \frac{1}{3x-2}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \boxed{\frac{1}{3}\ln|3x-2| + C}$$

Biff, you needed to make it the absolute value of $3x-2$ in order to avoid domain issues. Secondly, you need to add $(+C)$ to the end of an antiderivative in order to make it fit all regardless of the "C" value, also known as the most general antiderivative.

Good.

$\tan^2+1=\sec^2$

8. Derive the reduction formula

$$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$$

$\int \sec^2 u \cdot \sec^{n-2} u\,du$ $\quad w=\sec^{n-2} u \quad v=\tan u$

$w' = n-2\sec^{n-3} u \cdot \sec u \tan u \quad v'=\sec^2 u$

$\int \sec^n u\,du = \tan u \sec^{n-2} u - n-2\int \sec^{n-2} u \cdot \tan^2 u\,du$

$-n-2\int \sec^{n-2} u\,(\sec^2 u - 1)$

Nice!

$-n-2\left[\int \sec^n u\,du - \sec^{n-2} u\,du\right]$

$-n-2\int \sec^n u\,du + n-2\int \sec^{n-2} u\,du$

$+\,n-2\int \sec^n u\,du$

$\dfrac{n-1\int \sec^n u\,du = \tan u \sec^{n-2} u + n-2\int \sec^{n-2} u\,du}{n-1}$

$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$ ✓

9. It turns out there's a reason to care about $\int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\,dx$. Find the value of this integral.

Hey! It's improper! The denominator is 0 when $x = \pm r$!
Let's do the right half and use symmetry.

$$\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx = \lim_{b\to r^-} r\int_0^b \frac{1}{\sqrt{r^2-x^2}}\,dx$$

by Line 16

$$= \lim_{b\to r^-} r\cdot \sin^{-1}\frac{x}{r}\Big|_0^b$$

$$= \lim_{b\to r^-} r\cdot \sin^{-1}\frac{b}{r} - r\cdot\sin^{-1} 0$$

$$= \lim_{b\to r^-} r\cdot \sin^{-1}\frac{b}{r} - 0$$

$$= r\cdot \sin^{-1}\frac{r}{r}$$

$$= r\cdot \sin^{-1} 1$$

$$= r\cdot\frac{\pi}{2}$$

$$= \frac{\pi r}{2}$$

So the whole integral should be twice that, or

$$2\cdot\left(\frac{\pi r}{2}\right) = \pi\cdot r$$

10. Derive Line 23 from the Table of Integrals:

$$\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$$

Let $u = a\tan\theta$

$$\frac{du}{d\theta} = a\cdot\sec^2\theta$$

$$du = a\cdot\sec^2\theta\,d\theta$$

$$\int \frac{\sqrt{a^2+u^2}}{u}\,du = \int \frac{\sqrt{a^2+a^2\tan^2\theta}}{a\cdot\tan\theta}\cdot a\sec^2\theta\,d\theta$$

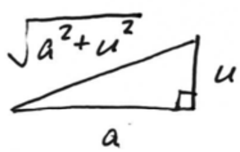
$$= \int \frac{\sqrt{a^2(1+\tan^2\theta)}}{a\tan\theta}\cdot a\sec^2\theta\,d\theta$$

$$= \frac{a}{a}\int \frac{a\sqrt{\sec^2\theta}\cdot\sec^2\theta\,d\theta}{\tan\theta}$$

$$= a\int \frac{\sec^3\theta}{\tan\theta}\,d\theta$$

$$= a\int \frac{(\tan^2\theta+1)\sec\theta\,d\theta}{\tan\theta}$$

$\tan\theta = \frac{u}{a}$



$$= a\int (\tan\theta + \cot\theta)\sec\theta\,d\theta$$

$$= a\int (\tan\theta\sec\theta + \csc\theta)\,d\theta$$

$$= a\cdot\sec\theta + a\cdot\ln|\csc\theta - \cot\theta| + C$$

$$= a\cdot\frac{\sqrt{a^2+u^2}}{a} - a\cdot\ln|\csc\theta + \cot\theta| + C$$

$$= \sqrt{a^2+u^2} - a\ln\left|\frac{\sqrt{a^2+u^2}}{u} + \frac{a}{u}\right| + C$$

Because

$$\frac{1}{\csc\theta - \cot\theta} = \frac{1}{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}}$$

$$= \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta}{1-\cos^2\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$