Exam 2 **Calculus 2** **3/2/18**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Set up an integral for the area of the region inside the circle with radius 1 centered at the origin and above the line $y = 1 - x$.

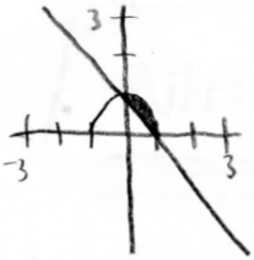
$r^2 = x^2 - y^2$

$y = \sqrt{r^2 - x^2}$

$y = \sqrt{1 - x^2}$

$$\int_0^1 \left[\sqrt{1 - x^2} - (1 - x)\right] dx$$

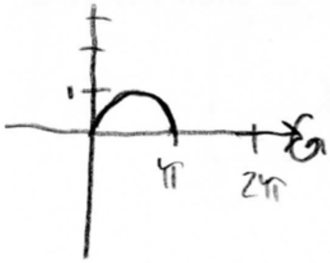
$$\boxed{\int_0^1 \left[\sqrt{1 - x^2} - 1 + x\right] dx}$$



Good

1

2. Set up an integral for the volume of the solid obtained when the region bounded between $y = \sin x$ and the x-axis is rotated around the x-axis.



$$\pi \int_0^{\pi} (\sin(x))^2 \, dx$$

$$\boxed{\pi \int_0^{\pi} \sin^2(x) \, dx}$$

Great

3. A force of 5 pounds is required to hold a spring stretched 0.6 feet beyond its natural length. How much work (in foot-pounds) is done in stretching the spring from its natural length to 0.7 feet beyond its natural length?

$F = kx$

$(5) = k(.6)$

$k = \frac{25}{3}$

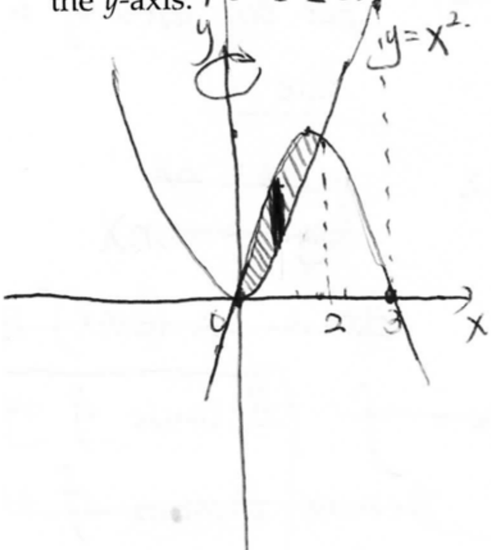
$W = \int_0^{.7} \left(\frac{25}{3}\right) x \, dx$

$W = \frac{25}{3} \cdot \frac{x^2}{2} \Big|_0^{.7}$

$\boxed{W = \frac{49}{24} \text{ ft} \cdot \text{lbs}}$

Great

4. Use the method of cylindrical shells to set up an integral for the volume of the solid obtained when the region bounded between $y = x^2$ and $y = 6x - 2x^2$ is rotated around the y-axis.



$y = x(6 - 2x)$

$y = 2x(3 - x)$

3×1.5

$x^2 = 6x - 2x^2$

$3x^2 = 6x$

$x^2 = 2x$

$x^2 - 2x = 0$

$x = 0, 2$

$$\text{Volume} = \int_0^2 2\pi x \cdot \left[(6x - 2x^2) - x^2 \right] dx$$

$$= \int_0^2 2\pi x \left(6x - 3x^2 \right) dx$$

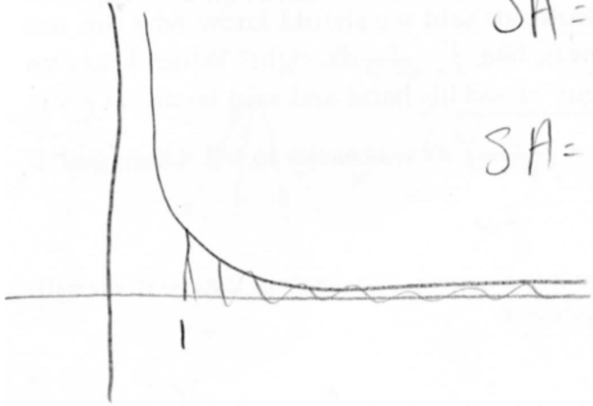
Good

$y = 1/x$

$y' = \frac{-1}{x^2}$

5. Consider the region below $y = \frac{1}{x}$, above the x-axis, and to the right of $x = 1$. Set up an integral to find the surface area of the solid obtained by rotating this region around the x-axis.

$$SA = \int_1^{\infty} 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$$

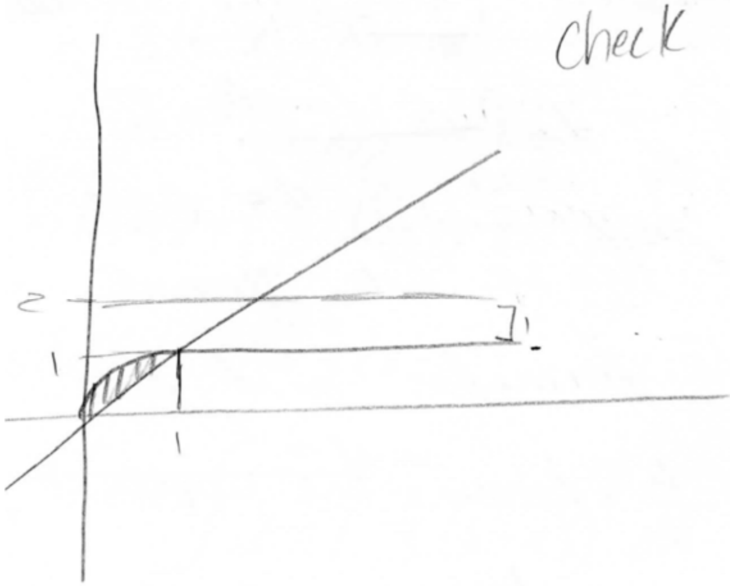


$$SA = \lim_{b \to \infty} \int_1^b 2\pi \cdot \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx$$

Great

6. Set up an integral for the volume of the solid obtained when the region bounded between $y = x$ and $y = \sqrt{x}$ is rotated around the line $y = 2$.

Check $2-1 = 1$ ✓



Great

$$V = \pi \int_0^1 (2-x)^2 - (2-\sqrt{x})^2\, dx$$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is so impossible! Every time I think I understand it, they totally do something different and I just can't even! So like, I was getting okay at the ones where you do areas and volumes and lengths and stuff, right? But then there was, like, the professor said we should know why this one integral was important, right? And it was, like, $\int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\,dx$, right? Which I have no idea why it was important, and this guy raised his hand and said he didn't get it, so the professor said, well, like $\int_{-r}^{r} \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$ was easier to tell. Ohmygod, is she totally insane?"

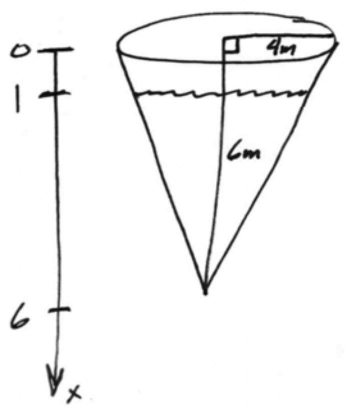
Help Bunny out by explaining what the two integrals she described have to do with each other, and why they might be important.

$\int_{-r}^{r} \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$ when simplified gives $\int_{-r}^{r} \frac{r}{\sqrt{r^2-x}}\,dx$ because $\int_{-r}^{r} \sqrt{\frac{r^2-x^2}{r^2-x^2}+\frac{x^2}{r^2-x^2}}\,dx$ is the same as the first with the inside of the parentheses squared and the 1 adjusted to find a common denomenator. adding them together gives $\int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\,dx$ and taking the square root provides the answer. This is important because it is the perimeter or arc length of a circle.

An arc length is equal to $\int_a^b \sqrt{1+(f'(x))^2}\,dx$ and the formula for a circle is $y=\sqrt{r^2-x^2}$ giving the derivative of $\frac{-x}{\sqrt{r^2-x^2}}$. Plugging this into the formula, we get the first equation $\int_{-r}^{r} \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$.

Excellent!

8. A tank in the shape of an inverted right circular cone has height 6 meters and radius 4 meters. It is filled with 5 meters of hot chocolate. Set up an integral for the work required to empty the tank by pumping the hot chocolate over the top of the tank. The density of hot chocolate is $\delta = 1040\, kg/m^3$.



Radius of a slice $= \left(4 - \frac{2}{3}x\right)$ m

Area of a slice $= \pi\left(4 - \frac{2}{3}x\right)^2$ m^2

Volume of slice $= \pi\left(4 - \frac{2}{3}x\right)^2 \Delta x$ m^3

Mass of slice $= \pi\left(4 - \frac{2}{3}x\right)^2 \cdot 1040\, \Delta x$ kg

Force for slice $= \pi\left(4 - \frac{2}{3}x\right)^2 \cdot 1040 \cdot 9.8\, \Delta x$ N

Work for slice $= \pi\left(4 - \frac{2}{3}x\right)^2 \cdot 10192 \cdot x\, \Delta x$ J

Total Work $= \int_1^6 \pi \cdot 10192\left(4 - \frac{2}{3}x\right)^2 x\, dx$ J

x	r
0	4
6	0

$$m = \frac{0-4}{6-0} = \frac{-2}{3}$$

$$r = \frac{-2}{3}x + 4$$

$\frac{1}{4}\int x\cos x - x\sin x$

$u = x \quad u' = 1$

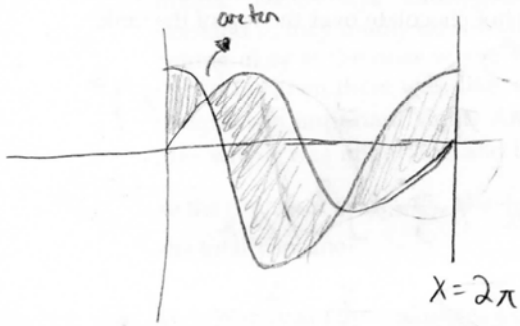
$v = \sin x \quad v' = \cos x$

$x\sin x + \cos x$

$(-\cos x) \quad (\sin x)$

$-x\cos x - \sin x$

9. Find the x-coordinate of the centroid of the region lying between the graphs of the functions $y = 9\sin x$ and $y = 9\cos x$ over the interval $[0, 2\pi]$.



$$X_{cm} = \frac{\int_a^b x \cdot f(x)\,dx}{\int_a^b f(x)\,dx}$$

$9\sin x = 9\cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \arctan(1)$

$$\int_0^{\arctan(1)} x\,(9\cos x - 9\sin x)\,dx$$

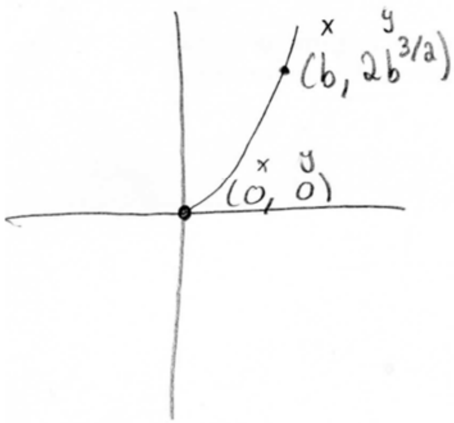
$$X_{cm} = \frac{\int_0^{\arctan(1)} x(9\cos x - 9\sin x)\,dx + \int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} x(9\sin x - 9\cos x)\,dx + \int_{\frac{5}{4}\pi}^{2\pi} x\,(9\cos - 9\sin x)\,dx}{\int_0^{\frac{1}{4}\pi}(9\cos x - 9\sin x)\,dx + \int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi}(9\sin x - 9\cos x)\,dx + \int_{\frac{5}{4}\pi}^{2\pi}(9\cos - 9\sin x)\,dx}$$

Excellent!

More on scatch paper where? ...

$$\frac{9(x\sin x + \cos x)\Big|_0^{\frac{1}{4}\pi} + 9(-x\cos x - \sin x)\Big|_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} + 9(x\sin + \cos x)\Big|_{\frac{5}{4}\pi}^{2\pi}}{9\sin x + 9\cos x\Big|_0^{\frac{1}{4}\pi} - 9\cos x - 9\sin x\Big|_{\frac{5}{4}\pi}^{2\pi} + 9\sin x + 9\cos x\Big|_{\frac{5}{4}\pi}}$$

10. Consider the curve $y = 2x^{3/2}$. Find a point $(b, 2b^{3/2})$ such that the length of the curve from $(0, 0)$ to $(b, 2b^{3/2})$ is exactly 16



$$16 = \int_0^b \sqrt{1 + 9x}\, dx$$

$u = 1 + 9x$

$du = 9\,dx$

$$16 = \frac{1}{9}\int_0^b u^{1/2}\, dx$$

$y' = 3x^{1/2}$

$(y')^2 = 9x$

$$16 = \frac{1}{9} \cdot \frac{2(1+9x)^{3/2}}{3} \Big|_0^b$$

$$16 = \frac{2(1+9(b))^{3/2}}{27} - \frac{2}{27}$$

$$\frac{434}{\cancel{27}} = \frac{2(1+9(b))^{3/2}}{\cancel{27}}$$

$$217 = (1 + 9b)^{3/2}$$

$$(217)^{2/3} = 1 + 9b$$

$$b = \frac{((217)^{2/3}) - 1}{9}$$

Yes!

$$\text{point} = \left(\frac{217^{2/3} - 1}{9},\ 2\left(\frac{217^{2/3} - 1}{9}\right)^{3/2}\right)$$