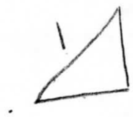
Each problem is worth 10 points. For full credit provide good justification for your answers.

1. (a) Convert $(1, \frac{\pi}{2})$ from polar to rectangular coordinates.

$x = 1 \cos \frac{\pi}{2} = 0$

$y = 1 \sin \frac{\pi}{2} = 1$ $\quad\quad \underline{(0,1)}$

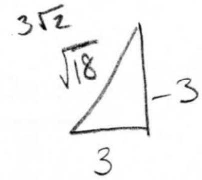


good

(b) Convert $(3, -3)$ from rectangular to polar coordinates.

$9 + 9 = c^2 \quad c = \sqrt{18}$ or $3\sqrt{2}$

$\tan\theta = \frac{-3}{3} = -1 \quad \theta = -\frac{\pi}{4}$



$\underline{(3\sqrt{2}, -\frac{\pi}{4})}$

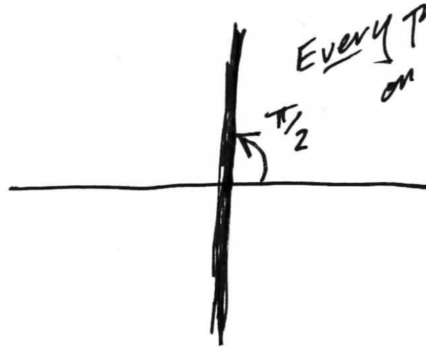
2. (a) Convert $x^2 + y^2 = 25$ to an equation in polar coordinates.

$r^2 = 25$

$r = 5$

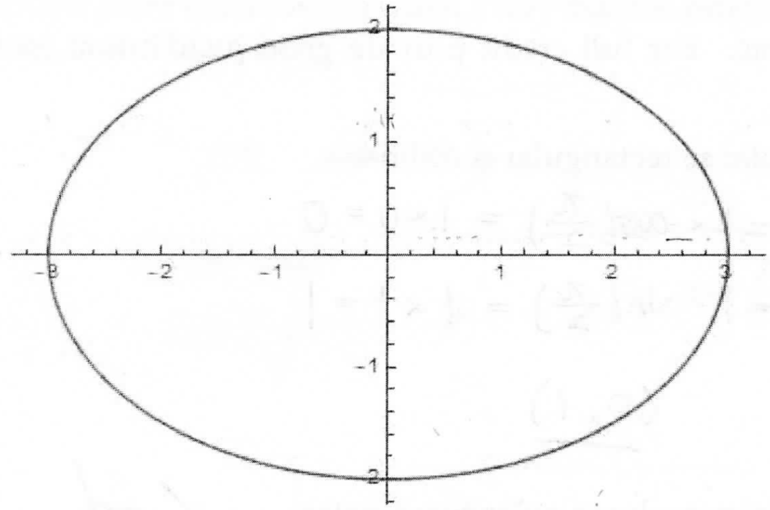
(b) Convert $\theta = \frac{\pi}{2}$ to an equation in rectangular coordinates.

$x = 0$



Every point on y-axis!

3. Write an equation for the conic section shown below:



This is an ellipse.

center $(0,0)$

long radius $= 3$

short radius $= 2$

Excellent

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

4. Set up an integral for the length of the curve $x = t-2\sin t, y = 1-2\cos t$ for $0 \le t \le 4\pi$.

$x = t - 2\sin t$

$y = 1 - 2\cos t$

$0 \le t \le 4\pi$

$\frac{dx}{dt} = 1 - 2\cos t$

$\frac{dy}{dt} = +2\sin t$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

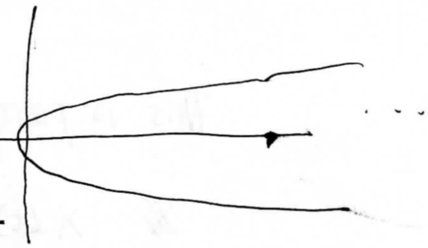
$$\boxed{\int_0^{4\pi} \sqrt{(1-2\cos t)^2 + (2\sin t)^2}\, dt}$$

Good

1/1

5. Find the exact coordinates of the leftmost point on the parametric curve with equations $x = t^4 - t^2, y = t + \ln(t)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 4t^3 - 2t$$



For the left points:

It's tangent is vertical

$$\therefore \frac{dy}{dx} = \frac{1 + \frac{1}{t}}{4t^3 - 2t}$$

Make $4t^3 - 2t = 0$

$$\therefore t(4t^2 - 2) = 0$$

$$2t(2t^2 - 1) = 0$$

$$2t(\sqrt{2}t + 1)(\sqrt{2}t - 1) = 0$$

$\therefore t = 0$ or $t = \frac{1}{\sqrt{2}}$ or $t = \frac{-1}{\sqrt{2}}$.

Because the domain of $\ln t$ is $(0, \infty)$

$\therefore t \neq 0$. and $t \neq \frac{-1}{\sqrt{2}}$

$\therefore$ Eventually $t = \frac{\sqrt{2}}{2}$

Excellent!

Plug in we get the coordinate:

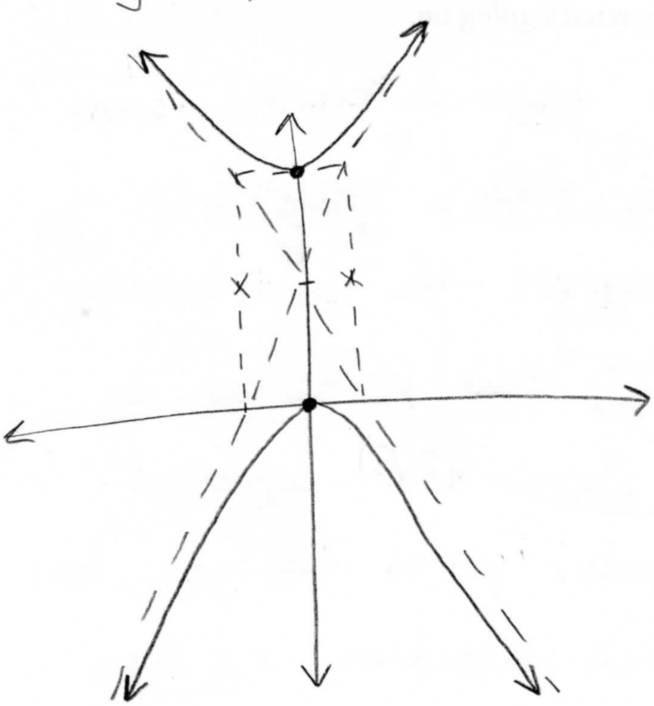
$$\left(\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{\sqrt{2}}{2}\right)^2, \frac{\sqrt{2}}{2} + \ln\left(\frac{\sqrt{2}}{2}\right) \right)$$

$$\Rightarrow \left(-\frac{1}{4}, \frac{\sqrt{2}}{2} + \ln\left(\frac{\sqrt{2}}{2}\right) \right)$$

6. Identify the graph of $y^2 - 8y = 16x^2$ as a parabola, hyperbola, or ellipse, give coordinates of its vertices, and sketch a decent graph.

$y^2 - 8y - 16x^2 = 0$ $\quad$ $y^2 - 8y + 16 - 16x^2 = 16$

$(y-4)^2 - 16x^2 = 16$ $\quad$ $\underline{\dfrac{(y-4)^2}{16} - x^2 = 1}$



vertices at

$\underline{(0,0)}$

$\underline{(0,8)}$

$\underline{\text{hyperbola}}$

$\underline{\text{Great}}$

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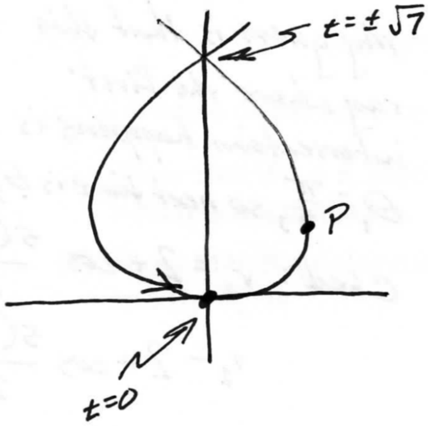
7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is so impossible! Every time I think I know something, they tell me it's not that simple anymore. So there was this question on the quiz, right? And it said to give three different pairs of coordinates for this point, right? But that's crazy, because, I mean, like if the coordinates for a point are $(2, 2)$ then they just *are*, right? If you give different coordinates then it's a different point, right? But the TA said something about how it's because of polar stuff, which I really don't get because there's trig, you know?"

Help Bunny out by explaining clearly what's going on.

Hey bunny so we learned this thing called polar cordinates and we had to tell a polar bear directions but the polar bear got really mad if you told him (2,2) because thats the long way so you had to convert it to polar $r^2 = 2^2 + 2^2 = 8$ $r = \sqrt{8}$ $\tan\theta = 1$ $\theta = \pi/4$ so the polar bear went to the angle $\pi/4$ + out $\sqrt{8}$, but then Jon got really tricky :) + told the bear to go to angle $5\pi/4$ + out $-\sqrt{8}$ and the bear ended up at the same spot because after you tell the bear an angle θ he is still at (0,0) waiting for distance so if you tell him a negative distance he will go backwards, also if you were to tell the bear $(-3\pi/4, -\sqrt{8})$ he would end up at the same place too!

Great!

8. Set up an integral or integrals for the area of the region enclosed by the loop of the curve with parametric equation $x = 7t - t^3, y = 8t^2$.



Crossover point when $x = 0$, so

$$0 = 7t - t^3$$

$$0 = t(7 - t^2)$$

$$t = 0 \text{ or } t = \pm\sqrt{7}$$

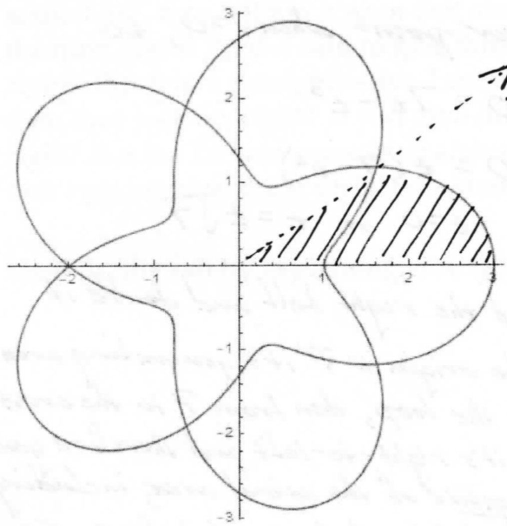
I'll find the right half and double it

From the origin to P it's generating area outside the loop, then from P to the crossing point it's right-to-left and therefore generating the negative of the actual area, including the part outside that we got before. So

$$-2\int_0^{\sqrt{7}} (8t^2)(7-3t^2)\,dt$$

is the area we want.

9. Set up an integral or integrals for the area of the region inside the curve with polar equation $r = 2 + \cos\frac{5\theta}{2}$.



My guess is that this ray where the first intersection happens is $\theta_1 = \frac{\pi}{5}$, so next time is $\theta_2 = \frac{\pi}{5} + 2\pi$.

Check: $r_1 = 2 + \cos\frac{5(\frac{\pi}{5})}{2} = 2$

$r_2 = 2 + \cos\frac{5(\frac{\pi}{5}+2\pi)}{2} = 2$

Yes! So each lobe is twice the shaded region, and there are five, so:

$$\text{Area} = 10 \cdot \frac{1}{2}\int_0^{\frac{\pi}{5}} \left(2 + \cos\frac{5\theta}{2}\right)^2 d\theta$$

10. [Anton 6th p. 712] In the late seventeenth century the Italian astronomer Giovanni Domenico Cassini (1625-1712) introduced the family of curves

$$(x^2 + y^2 + a^2)^2 - b^4 - 4a^2x^2 = 0 \quad (a > 0, b > 0)$$

in his studies of the relative motions of the Earth and the Sun.

(a) Show that if $a = b$, then the polar equation of the Cassini oval is $r^2 = 2a^2 \cos 2\theta$, which is a lemniscate.

(b) Show that in a polar coordinate system the distance d between the points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

a) Substitute a for b:

$$(x^2+y^2+a^2)^2 - a^4 - 4a^2x^2 = 0$$

Convert to polar:

$$(r^2+a^2)^2 - a^4 - 4a^2r^2\cos^2\theta = 0$$

$$r^4 + 2r^2a^2 + a^4 - a^4 - 4a^2r^2\cos^2\theta = 0$$

$$r^2(r^2 + 2a^2 - 4a^2\cos^2\theta) = 0$$

So for $r \neq 0$,

$$r^2 + 2a^2 - 4a^2\cos^2\theta = 0$$

$$r^2 = 2a^2(2\cos^2\theta - 1)$$

So by Double-angle for cos,

$$r^2 = 2a^2\cos 2\theta \quad \square$$

b) So $x_1 = r_1\cos\theta_1$, $y_1 = r_1\sin\theta_1$, $x_2 = r_2\cos\theta_2$, $y_2 = r_2\sin\theta_2$:

$$d = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)}$$

$$= \sqrt{r_1^2\cos^2\theta_1 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos^2\theta_2 + \ldots}$$

$$\overline{r_1^2\sin^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin^2\theta_2}$$

$$= \sqrt{r_1^2(\cos^2\theta + \sin^2\theta) + r_2^2(\cos^2\theta + \sin^2\theta) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

(subtraction identity for cos)

$\square$