Examlet 3 **Foundations of Advanced Math** **3/23/18**

1. (a) State the definition of an injective function.

We say f is an injective function iff $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

(b) Give an example of an injective function.

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is injective since

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

(c) Give an example of a function which is not injective.

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5$ is not injective since

$$f(2) = 5 = f(3), \text{ but } 2 \neq 3.$$

2. (a) The sum of two odd functions, both with domain $\mathbb{R}$, is odd.

Proof: $f(x): \mathbb{R} \to \mathbb{R}$ and $g(x): \mathbb{R} \to \mathbb{R}$ are both odd functions.
By the definition of an odd function, f and g are odd iff $-f(x) = f(-x)$ and $-g(x) = g(-x)$.
By the definition of adding functions, $(f+g)(-x) = f(-x) + g(-x)$.
Since $f(x)$ and $g(x)$ are odd, $f(-x) + g(-x) = -f(x) + (-g(x))$.
We can factor out (-1) to get $-f(x) + -(g(x)) = -(f(x) + g(x)) = -(f+g)(x)$.
$(f+g)(-x) = -(f+g)(x)$, so by definition, $(f+g)(x)$ is an odd function. $\square$

Good

(b) The composition of two odd functions, both with domain $\mathbb{R}$, is even.

Counterexample: $f(x) = x$ and $g(x) = x^3$ are both odd functions.
$f \circ g(x) = f(g(x)) = x^3$.
At $x = 1$, $f \circ g(x) = 1$
$f \circ g(-x) = -1$.
$1 \neq -1$, so $f \circ g(x) \neq f \circ g(-x)$.
For $f \circ g(x)$ to be even, $f \circ g(x)$ would have to equal $f \circ g(-x)$, so $f \circ g(x)$ is not even. $\square$

Excellent!

Counterproof: Take two odd functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. f and g are odd iff $f(-x) = -f(x)$ and $g(-x) = -g(x)$.
By the definition of composition of functions,
$f \circ g(-x) = f(g(-x))$
Because g is odd,
$f(g(-x)) = f(-g(x))$
Because f is also odd
$f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$
Therefore, $(f \circ g)(-x) = -(f \circ g)(x)$.
By definition, $f \circ g(x)$ is an odd function.
>At this point, I realize that just because $(f \circ g)(x)$ is odd doesn't technically mean it's not even, I've included a counterexample.

3. If $f : A \to B$ and $g : B \to C$ are surjective functions, then $g \circ f$ is surjective.

Take $c \in C$, so since g is surjective $\exists b \in B$ for which $g(b) = c$. Now since f is surjective, for this $b \in B$ we know $\exists a \in A$ for which $f(a) = b$. But then $g \circ f(a) = g(f(a)) = g(b) = c$, for for any $c \in C$ we can find $a \in A$ for which $g \circ f(a) = c$, so $g \circ f$ is surjective. $\square$

4. Let $f : A \to B$ be a bijective function. Then the inverse function of f is unique, i.e. if g_1 and g_2 are both inverse functions for f, then $g_1 = g_2$.

S'pose $f : A \to B$ is bijective and both g_1 and g_2 are inverse functions for f. Then we know $g_1 : B \to A$ and $g_2 \; B \to A$. Since f is surjective, $\forall b \in B \; \exists a \in A$ such that $f(a) = b$, so $g_1(f(a)) = g_1(b)$ or $a = g_1(b)$, and $g_2(f(a)) = g_2(b)$ so $a = g_2(b)$. But then $g_1(b) = g_2(b)$ for every $b \in B$, and thus $g_1 = g_2$. $\square$

5. $\mathbb{N}$ is equipollent to $\mathbb{Z}$.

for $\mathbb{N}$ to be equipollent to $\mathbb{Z}$ there must exist a bijection from $\mathbb{N} \to \mathbb{Z}$.

$$f(n) = \begin{cases} -\left(\frac{n}{2}+1\right) \text{ if } n \text{ is even.} \\ \frac{n-1}{2} \text{ if } n \text{ is odd} \end{cases} \quad \text{where } n \in \mathbb{N}$$

Excellent

$-\left(\frac{0}{2}+1\right)=-1$, $-\left(\frac{2}{2}+1\right)=-2$, ... ✓

$\left(\frac{1-1}{2}\right)=0$

$\frac{3-1}{2}=1$

... ✓

when n is even, the integers -1, -2, -3 ... are produced

when n is odd, the integers 0, 1, 2 ... are produced

$f: \mathbb{N} \to \mathbb{Z}$ is bijective.