Examlet 4 **Foundations of Advanced Math** **4/16/18**

1. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b \Leftrightarrow 3|(b-a)$.

   (a) Determine whether $\sim$ is reflexive.

   S'pose $a \in \mathbb{Z}$.
   $a \sim a \iff 3|(a-a) \iff 3|0$, but $3|0$ is true since $0 \in \mathbb{Z}$ and $3 \cdot 0 = 0$. So $a \sim a$, and the relation is reflexive.

   (b) Determine whether $\sim$ is symmetric.

   S'pose $a, b \in \mathbb{Z}$ and $a \sim b$.
   Then $3|(b-a)$, so $\exists n \in \mathbb{Z}$ for which $3n = b-a$. But then $a-b = 3(-n)$, and $-n \in \mathbb{Z}$ by closure, so $b \sim a$ and the relation is symmetric

   (c) Determine whether $\sim$ is transitive.

   S'pose $a, b, c \in \mathbb{Z}$ and $a \sim b \wedge b \sim c$.
   Then $3|(b-a)$ and $3|(c-b)$, so $\exists m, n \in \mathbb{Z}$ such that $b-a = 3m$ and $c-b = 3n$. Adding these equations, $c-a = 3m+3n = 3(m+n)$. Since $m+n \in \mathbb{Z}$ by closure, $3|(c-a)$ and thus $a \sim c$. Therefore the relation $\sim$ is transitive.

2. Consider the relation on $\mathbb{N}$ defined by $a \approx b \Leftrightarrow b = n \cdot a$ for some $n \in \mathbb{N}$.

(a) Determine whether $\approx$ is reflexive.

For $a \approx a$ to be true, then there must be some $n \in \mathbb{N}$ that allows $a = n \cdot a$. And as 1 is an element of $\mathbb{N}$, we can say $a = 1 \cdot a = a$.

Good!

So because $a = 1 \cdot a$ is true we can say $a \approx a$, making it reflexive. $\square$

(b) Determine whether $\approx$ is symmetric.

Though $1 \approx 2$ because $2 = n \cdot 1$ where $n = 2$

$2 \not\approx 1$ because there is no $n \in \mathbb{N}$ such that $1 = n \cdot 2$,

Making it not symmetric. $\square$ Great

(c) Determine whether $\approx$ is transitive.

If $a \approx b$ then $b = n \cdot a$ for some $n \in \mathbb{N}$

and if $b \approx c$ then $c = m \cdot b$ for some $m \in \mathbb{N}$.

We can say $c = m \cdot (n \cdot a)$ by substituting b.

then we know $m \cdot n$ is an element of $\mathbb{N}$ because they are each elements of $\mathbb{N}$, allowing $a \approx c$, making it transitive. $\square$

Great

3. Consider the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,5),(4,4),(5,3),(5,5)\}$. Give the equivalence classes of R and the partition associated with R.

$\hookrightarrow \Pi$

$[1] = \{1,2\}$

$[2] = \{1,2\}$

$[3] = \{3,5\}$

$[4] = \{4\}$

$[5] = \{3,5\}$

$\Pi = \{\{1,2\},\{3,5\},\{4\}\}$

Great

4. We say that two vertices v_1 and v_2 of a graph G are **on a common cycle of** $G \Leftrightarrow \exists$ a cycle including v_1 and v_2.

(a) The relation of being on a common cycle of a graph is reflexive.

False. In this graph:

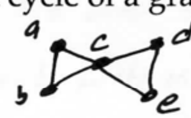
there are no cycles, so neither vertex is on a common cycle with itself.

(b) The relation of being on a common cycle of a graph is symmetric.

True. If v_1 is on a common cycle with v_2, then it's also the case that the very same cycle contains v_2 and v_1.

(c) The relation of being on a common cycle of a graph is transitive.

False. In this graph:



we have a and c on a common cycle, and we have c and d on a common cycle, but a and d are not on a common cycle because vertex c would need to be used twice.

5. The number of edges in a tree with n vertices is $\underline{\quad n-1 \quad}$.
[Yes, you need to justify your answer.]

Let's use induction!
Base Case: If $n=1$, the tree has no edges, and $1-1=0$.

Tree with $n=1$

Inductive Step: S'pose any tree with n vertices has $n-1$ edges, and consider a tree with $n+1$ vertices. By another exercise, this tree must have at least one vertex with degree 1. Consider the graph that results when this vertex and its adjacent edge are removed. The resulting graph is still a tree, since removing a vertex of degree 1 keeps a connected graph connected. Thus as a tree with $(n+1)-1=n$ vertices, it must have $n-1$ edges by hypothesis. But returning our new vertex and edge adds one edge (no more or we'd create a cycle), for $(n-1)+1=n$ edges.

Thus we have a base case, and if it's true for n vertices it's true for $n+1$ vertices, so by induction it's true for any tree. $\square$