Exam 1 **Calculus 2** **2/6/19**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Evaluate

$$\int \frac{1}{x^2+4}\,dx$$

Line 17

$a = 2$, $u = x$

$$\boxed{= \frac{1}{2}\tan^{-1}\frac{x}{2} + C}$$

Great!

2. Evaluate

$$\int x e^x \, dx$$

integration by parts

$\int u \cdot v' = u \cdot v - \int u' \cdot v$

$\int x e^x dx$

$u = x \quad u' = 1$

$v = e^x \quad v' = e^x$

$\int x e^x dx = x e^x - \int 1 e^x \, dx$

$\int x e^x dx = x e^x - 1 \int e^x \, dx$

$$\boxed{\int x e^x dx = x e^x - e^x + C}$$

Great

3. Evaluate

$$\int \sqrt{4-2x}\,dx$$

Let $u = 4-2x$

$\frac{du}{dx} = -2$

$dx = \frac{du}{-2}$

$\int \sqrt{u}\ \frac{du}{-2} = -\frac{1}{2}\int u^{1/2}\,du$

$\frac{2}{3}u^{3/2}$

$= \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C$

$= \boxed{\left(-\frac{1}{2}\right)\frac{2}{3}(4-2x)^{3/2} + C}$

Excellent!

4. Evaluate

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx$$

$\frac{5X+1}{(2X+1)(X-1)} = \frac{A}{(2X+1)} + \frac{B}{(X-1)}$

$[(2X+1)(X-1)]\frac{5X+1}{(2X+1)(X-1)} = \left[\frac{A}{2X+1} + \frac{B}{X-1}\right][(2X+1)(X-1)]$

$5X+1 = \frac{A\cancel{(2X+1)}(X-1)}{\cancel{2X+1}} + \frac{B(2X+1)\cancel{(X-1)}}{\cancel{X-1}}$

Very Nice!

$5X+1 = A(X-1) + B(2X+1)$

$5X+1 = AX - A + 2BX + B$

for X: $5 = A + 2B \rightarrow 5 = A + 2(1+A) \rightarrow 5 = A + 2 + 2A$

$3 = 3A$

constants: $1 = -A + B \rightarrow B = 1 + A$

$B = 1 + 1 \longrightarrow \left\{\begin{array}{l} A = 1 \\ B = 2 \end{array}\right\}$

$\int \frac{A}{2X+1} + \frac{B}{X-1} = \int \frac{1}{2X+1}dx + \int \frac{2}{X-1}dx$

$u = 2X+1$, $\frac{du}{dx} = 2$, $\frac{du}{2} = dx$

$v = X-1$, $\frac{dv}{dx} = 1$, $dv = dx$

$= \int \frac{1}{u}\left(\frac{du}{2}\right) + 2\int \frac{1}{X-1}dv$

$= \frac{1}{2}\int \frac{1}{u}du + 2\int \frac{1}{v}dv = \boxed{\frac{1}{2}\ln|2X+1| + 2\ln|X-1| + C}$

5. Evaluate

$$\int \tan^7\theta \sec^2\theta \, d\theta$$

let $u = \tan(\theta)$

$\frac{du}{d\theta} = \sec^2(\theta)$

$du = \sec^2(\theta)\, d\theta$

$\int u^7 \, du$

$\frac{1}{8} u^8 + C$

$\boxed{\frac{1}{8}\tan^8(\theta) + C}$

Great!

6. Evaluate

$$\int_2^{\infty} e^{-5t} \, dt$$

$\lim_{b \to \infty} \int_2^b e^{-5t} \, dt$

$\lim_{b \to \infty} -\frac{1}{5} e^{-5t} \Big|_2^b$

$\lim_{b \to \infty} -\frac{1}{5} e^{-5b} + \frac{1}{5} e^{-5(2)}$

$\lim_{b \to \infty} -\frac{1}{5e^{5b}} + \frac{1}{5e^{10}}$

approaches $\emptyset$

$\boxed{\frac{1}{5e^{10}}}$

Nice!

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, Calc is tough! I thought I was pretty good, but it's like you actually have to study this stuff or something! Unbelievable! So especially this trig sub stuff is killing me. I mean, how am I supposed to know when you do tan and when you do sin and stuff? Is it just magic?"

   Help Biff out by explaining how to tell when to use which trig substitution.

Hey Biff, there are two really useful identities that you should know — and the right trig substitution depends on which identity you want to use:

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

So, basic rule of thumb: LOOK under the square root. If its something square PLUS x^2, then use tan, because that will make it look like the second identity. If its $a^2 - x^2$, use sin because you can make it look like the first identity ($1 - \sin^2 x = \cos^2 x$), And if it's $x^2 - a^2$, use sec because you can make it look like $\sec^2 x - 1 = \tan^2 x$. Either way, the goal is to get something squared under the square root b/c that simplifies nicely.

[Def. go see Jon if you're not sure b/c he explained this to me.] ☺

8. It turns out there's a reason to care about $\int_0^r \sqrt{r^2 - x^2}\,dx$. Find the value of this integral.

$$\int \sqrt{r^2-x^2}\,dx = \int \sqrt{r^2 - r^2\sin^2\theta}\cdot r\cos\theta\,d\theta \qquad \text{Let } x = r\sin\theta$$

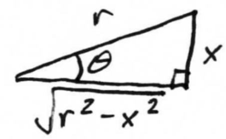
$$= \int r\sqrt{1-\sin^2\theta}\cdot r\cos\theta\,d\theta \qquad \frac{dx}{d\theta} = r\cos\theta$$

By Line C4 (or below)

$$= r^2 \int \cos^2\theta\,d\theta \qquad dx = r\cos\theta\,d\theta$$

$$= r^2\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)$$

$$= r^2\left(\frac{\theta}{2} + \frac{1}{4}\cdot 2\sin\theta\cos\theta\right)$$



$$= r^2\left(\frac{1}{2}\arcsin\frac{x}{r} + \frac{1}{2}\cdot\frac{x}{r}\cdot\frac{\sqrt{r^2-x^2}}{r}\right)\Big|_0^r$$

$$= r^2\left[\left(\frac{1}{2}\arcsin 1 + \frac{1}{2}\cdot 1\cdot 0\right) - \left(\frac{1}{2}\cdot\arcsin 0 + \frac{1}{2}\cdot\frac{0}{r}\cdot\frac{r}{r}\right)\right]$$

$$= \frac{r^2}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi r^2}{4}$$

Parts!

$u = \cos\theta \quad v = \sin\theta$

$u' = -\sin\theta \quad v' = \cos\theta$

$$\int \cos^2\theta\,d\theta = \cos\theta\sin\theta + \int \sin^2\theta$$

$$= \sin\theta\cos\theta + \int (1-\cos^2\theta)\,d\theta$$

$$\int \cos^2\theta\,d\theta = \sin\theta\cos\theta + \theta - \int\cos^2\theta\,d\theta$$

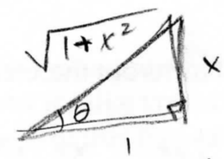
Add $\int\cos^2\theta\,d\theta$ to both sides

$$2\int\cos^2\theta\,d\theta = \sin\theta\cos\theta + \theta + C$$

$$\int\cos^2\theta\,d\theta = \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C$$

9. Evaluate

$$\int_0^1 \tan^{-1} x \, dx$$



$\theta = \tan^{-1} x$

Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$= \int_{0=x}^{1=x} \tan^{-1}(\tan\theta) \sec^2\theta \, d\theta$$

$$= \int_{0=x}^{1=x} \theta \sec^2\theta \, d\theta \longrightarrow \text{Now integrate by parts.}$$

$u = \theta \qquad v = \tan\theta$

$u' = d\theta \qquad v' = \sec^2\theta$

$$= \theta\tan\theta \Big|_{x=0}^{x=1} - \underbrace{\int_{x=0}^{x=1} \tan\theta \, d\theta}_{\text{by line 12}}$$

$$= \theta\tan\theta - \ln|\sec\theta| \Big|_{x=0}^{x=1}$$

Now sub. x back in.
Use the triangle to find sec θ.

$$= (\tan^{-1}x)(\tan(\tan^{-1}x)) - \ln\left|\frac{\sqrt{1+x^2}}{1}\right| \Bigg|_{x=0}^{x=1}$$

this is an exponent which can come down

Outstanding!

$$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| \Big|_{x=0}^{x=1}$$

$$= \left(1\tan^{-1}(1) - \frac{1}{2}\ln|2|\right) + \left(-0 + \frac{1}{2}\ln|1|\right)$$

$$= \boxed{\frac{\pi}{4} - \frac{\ln 2}{2} + \frac{\ln 1}{2}}$$

10. Derive Line 54 from the Table of Integrals,

$$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$$

$\frac{2u}{3b}(a+bu)^{3/2} - \frac{2}{3b}\int (a+bu)^{3/2}\,du$

$f = u$ $\quad g = \frac{2}{3b}(a+bu)^{3/2}$

$f' = 1$ $\quad g' = (a+bu)^{1/2}$

$= -\frac{2}{3b^2}\int V^{3/2}\,dV$

V-sub ← B/c u is taken

$V = a+bu$

$dv = b\,du$

$= \left(-\frac{2}{3b^2}\right)\left(\frac{2}{5}V^{5/2}\right)+C$

$= \frac{2u}{3b}(a+bu)^{3/2} - \left(\frac{2}{3b^2}\right)\left(\frac{2}{5}V^{5/2}\right)+C$ $\quad du = \frac{1}{b}$

$= \frac{2u}{3b}(a+bu)^{3/2} - \left(\frac{4}{15b^2}\right)(a+bu)^{5/2}+C$?

$= \frac{2u}{3b}(a+bu)^{3/2} - \left(\frac{4}{15b^2}\right)(a+bu)(a+bu)^{3/2}+C$

$\left[\frac{2u}{3b} - \frac{4}{15b^2}(a+bu)\right](a+bu)^{3/2}+C$

$\frac{10ub}{15b^2} - \left(\frac{4a+4bu}{15b^2}\right)$

Excellent!

$= \frac{-4a+6bu}{15b^2}$

$$\boxed{\frac{2}{15b^2}(-2a+3bu)(a+bu)^{3/2}+C}$$