Examlet 1 **Foundations of Advanced Math** **2/8/19**

1. The product of two throddodd integers is throdd.

Proof: A throddodd integer can be expressed as $3x+2$ for some $x \in \mathbb{Z}$, so let's say we have

$$m = 3x+2 \quad \text{and} \quad n = 3y+2, \quad x, y \in \mathbb{Z}.$$

Then the product

$$\begin{aligned} m \cdot n &= (3x+2)(3y+2) \\ &= 9xy + 6x + 6y + 4 \\ &= 9xy + 6x + 6y + 3 + 1 \\ &= 3(3xy + 2x + 2y + 1) + 1. \end{aligned}$$

Since $3xy + 2x + 2y + 1$ is an integer by closure, $m \cdot n$ is three times an integer plus 1, and therefore by definition,

THRODD. $\square$

Excellent!

2. Show that if $n, s, t \in \mathbb{Z}$ with $n|s$ and $n|(s+t)$, then $n|t$.

rewrite as

$s = nx$ where $x \in \mathbb{Z}$

rewrite as

$s + t = ny$ where $y \in \mathbb{Z}$

Prove $\rightarrow$ $t = nz$ where $z \in \mathbb{Z}$

Substitute nx for s in the second equation.

$nx + t = ny$

When we rearrange, we find

$t = ny - nx$

Factor out an n

$t = n(y-x)$

Since $y-x \in \mathbb{Z}$ by closure of integers, we know that $n \times$ (some integer) $= t$, which means $n|t$ by definition.

Good

3. Determine whether the statements $(P \Rightarrow Q)$ and $(\neg P \vee Q)$ are logically equivalent.

P	Q	$P \Rightarrow Q$	$\neg P$	$(\neg P \vee Q)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Proof: Well, since the truth values are the same under ALL circumstances, $P \Rightarrow Q$ and $(\neg P \vee Q)$ are logically equivalent. □

Great!

4. Use induction to show that the sum of any two consecutive natural numbers is odd.

Let's use induction with a base case of $n=0$.

then $0+1=1=2(0)+1$, which is odd, so our base case works.

Let's assume this works for any $n=k$ such that $k+(k+1)=2n+1$ for $n \in \mathbb{Z}$.

we want to show this works for $k+1$ such that $(k+1)+(k+2)=2m+1$ for $m \in \mathbb{Z}$.

$$k+(k+1)=2n+1$$
$$k+(k+1)+2=2n+1+2$$
$$k+1+k+2=2n+2+1$$
$$(k+1)+(k+2)=2(n+1)+1$$

Because $n+1$ is an int by closure, call it $m \in \mathbb{Z}$, $(k+1)+(k+2)=2m+1$, which is what we wanted to prove.

therefore, by induction, for any $n=k$ or $n=k+1$ int ints, $n+(n+1)$ is odd, so the sum of any two natural numbers is odd. $\square$

Great!

5. $\sqrt{2}$ is irrational.

Proof

let $\sqrt{2}$ be not irrational.

then $\sqrt{2}$ is rational + can be expressed as

$\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and have no common factors

$2 = \frac{a^2}{b^2}$

$2b^2 = a^2$, since b^2 is an integer by closure, $2b^2$ is even, which means a^2 is even. and because the only number that squares to an even is an even, a is even as well + can be expressed as $a = 2p$ for $p \in \mathbb{Z}$

So: $2b^2 = (2p)^2$

$2b^2 = 4p^2$

$b^2 = 2p^2$, for the same reason, p^2 is an integer by closure so $2p^2$ is even, which means b^2 is even and also b. b could then be expressed as $b = 2q$, but that would mean a + b have a common factor of 2, which contradicts our earlier statement if $\sqrt{2}$ were to be rational, which means $\sqrt{2}$ is not rational, and indeed irrational. $\square$

Good