Examlet 4 **Foundations of Advanced Math** **4/16/18**

1. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b \Leftrightarrow a - b$ is odd.

   (a) Determine whether and why $\sim$ is reflexive.

Proof S'pose $\sim$ is reflexive. $\forall a \in \mathbb{Z}$ $a - a = 0 = 2n+1$ for some $n \in \mathbb{Z}$

Then $n = -\frac{1}{2} \in \mathbb{Z}$

Then by the closure of integers for multiplication, $n(-1) - \frac{1}{2} = \frac{1}{2} \in \mathbb{Z}$

Assuming $\frac{1}{2}$ is not an integer, our supposition is not true and thus $\sim$ is not reflexive. $\square$

Good.

   (b) Determine whether and why $\sim$ is symmetric.

Proof S'pose $a \sim b$.

Then $a - b = 2n+1$ for some $n \in \mathbb{Z}$

Then $-(a-b) = -(2n+1) = -a+b = -2n-1 = b-a = 2(-n+1)+1$ for some $n \in \mathbb{Z}$

As $n, -1 \in \mathbb{Z}$, $-n+1 \in \mathbb{Z}$ by the closure of the integers for addition and multiplication

Then $b - a = 2m+1$ for some $m \in \mathbb{Z}$ $(m = -n-1)$

Then $b - a$ is odd.

Then $b \sim a$.

Then $a \sim b \Rightarrow b \sim a$ an thus the relation is symmetric $\square$

Good

   (c) Determine whether and why $\sim$ is transitive.

CounterExample: $1 \sim 0$ $(1 - 0 = 1 = 2(0)+1$ and $0 \in \mathbb{Z}$ by definition$)$.

$0 \sim 1$ $(0 - 1 = -1 = 2(-1)+1$ and $-1 \in \mathbb{Z}$ by definition$)$.

$1 - 1 = 0$ which assuming $\frac{1}{2}$ is not an integer, as above, is not odd

$1 \not\sim 1$

$\therefore$, $a \sim b \wedge b \sim c \not\Rightarrow a \sim c$ and thus the relation is not transitive

Nice.

2. Consider the relation on some collection of sets defined by $A \approx B \Leftrightarrow \exists$ a bijection $f : A \to B$.

(a) Determine whether and why $\approx$ is reflexive.

$A \approx A$ iff $\exists$ a bijection $f: A \to A$.

The identity function is a bijection $f: A \to A$.

Therefore, $A \approx A$, so the relation is reflexive.

Excellent!

(b) Determine whether and why $\approx$ is symmetric.

Spose $A \approx B$, so $\exists$ a bijection $f: A \to B$.

We know that bijections have inverses that are also bijections, so $\exists$ a bijection $f^{-1}: B \to A$.

Great! So $B \approx A$, and the relation is symmetric.

(c) Determine whether and why $\approx$ is transitive.

Spose $A \approx B$ and $B \approx C$, so $\exists$ a bijection $f: A \to B$ and a bijection $g: B \to C$.

We know the composition of two bijections is also a bijection, so $\exists$ a bijection $g \circ f : A \to C$.

So $A \approx C$ and the relation is transitive

$\forall b \in B, \exists a \in A \ni f(a) = b$

$\forall (a_1, b_1)(a_2, b_2) \in f,\ b_1 = b_2 \Rightarrow a_1 = a_2$

$\forall c \in C, \exists b \in B \ni g(b) = c$ *Hm. Same b?*

$\forall (b_1, c_1)(b_2, c_2) \in g,\ c_1 = c_2 \Rightarrow b_1 = b_2$.

$\Longrightarrow$

$\forall c \in C, \exists a \in A \ni g(f(a)) = c$

$\forall (a_1, c_1)(a_2, c_2) \in g \circ f,$

$c_1 = c_2 \Rightarrow a_1 = a_2$

$g \circ f$ is surjective + injective

$\therefore$ bijective.

3. Let $S = \{a, b, c, d\}$, and that $\sim = \{(a,a), (b,b), (b,c), (c,b), (c,c), (d,d)\}$.

(a) Give the equivalence classes of $\sim$.

$[a] = \{a\}$

$[b] = \{b, c\}$

$[c] = \{b, c\}$

$[d] = \{d\}$

These are the equivalence classes of $\sim$.

Great

(b) Give the partition associated with $\sim$.

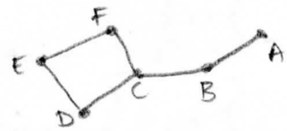
Let Π be a partition associated with $\sim$.

then $\Pi = \{\{a\}, \{b, c\}, \{d\}\}$.

4. Suppose that G is a graph with at least one cycle. We say that two vertices v_1 and v_2 of a graph G are **on a common cycle of** $G \Leftrightarrow \exists$ a cycle including v_1 and v_2.

(a) The relation of being on a common cycle of a graph is reflexive.

False.



Consider the graph:

Excellent!

G has at least one cycle, but A and A are not on a common cycle, because there is no cycle with distinct vertices starting and ending at A.

(b) The relation of being on a common cycle of a graph is symmetric.

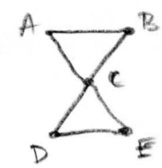
Spose V_1 and V_2 are on a common cycle. $(V_1 \sim V_2)$
Then there exists a cycle including V_1 and V_2.
The very same cycle includes V_2 and V_1, so V_2 and V_1 are on a common cycle. $(V_2 \sim V_1)$

Great

$\therefore$ The relation is symmetric.

(c) The relation of being on a common cycle of a graph is transitive.

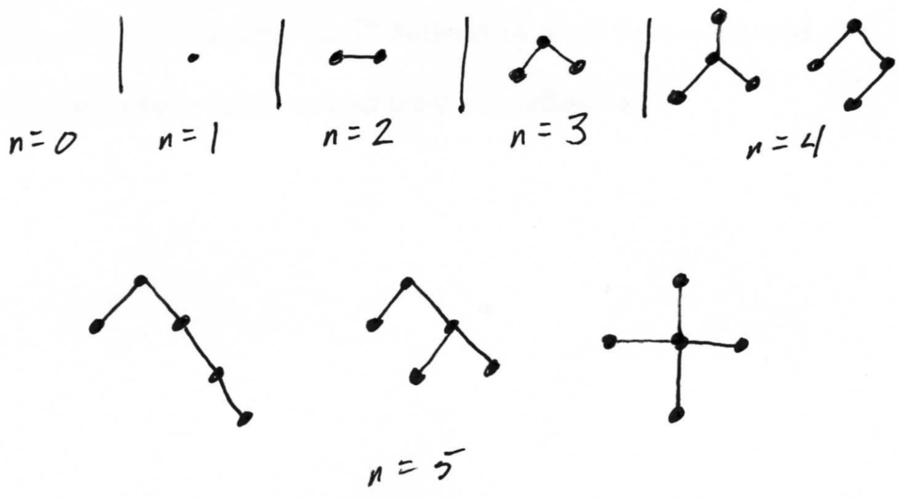
False. Consider the following graph:



A and C are on a common cycle and C and E are on a common cycle, but A and E are not on a common cycle because there is no cycle including A and E that crosses C only once.

Nice

5. (a) Give all trees with $n \le 5$ vertices.



(b) The minimum number of vertices with degree 1 in a tree with n vertices is

$\underline{2 \text{ for } n \geq 2,\ 0 \text{ for } n < 2}$.

The examples above suffice for $n=0$ and $n=1$.

Then induct, using $n=2$ as a base case, where there are at most two vertices of degree 1 as shown above.

So spose it's true for some $n=k$ vertices, with $n \geq 2$. Consider a tree with $k+1$ vertices. One possibility is a single chain with two endpoints of degree 1 and a string of degree 2 vertices between them, so the minimum cannot be more than 2. Now spose our tree with $k+1$ vertices has exactly 1 vertex of degree 1; then removing it and the edge adjoining it leaves a tree with k vertices and 0 or 1 vertices of degree 1, contradicting our hypothesis. Now instead suppose our tree with $k+1$ vertices has exactly 0 vertices of degree 1; then all vertices have at least degree 2. Remove one, along with its adjoining edges. We now have a tree with k vertices, so two of the vertices that used to be adjacent to our removed vertex must be connected by a walk. But then we had a cycle in our tree with $k+1$ vertices, another contradiction. □