Examlet 1A **Foundations of Advanced Math** **2/15/21**

1. The product of any two odd integers is odd.

Let odd integer $x = 2a+1$ and odd integer $y = 2b+1$ where $a, b \in \mathbb{Z}$.

Then, $x \cdot y = (2a+1)(2b+1) = 4ab+2a+2b+1$

$4ab+2a+2b+1 = 2(2ab+a+b)+1$

$(2ab+a+b)$ is an integer by closure, so the product of x and y is odd $(2m+1)$. $\square$

Great

2. Show that if $p \in \mathbb{Z}$ and $5|p^2$ then $5|p$.

Well, as long as we know that for any $n \in \mathbb{Z}$ exactly one of the following is true:

$n \equiv_5 0$
$n \equiv_5 1$
$n \equiv_5 2$
$n \equiv_5 3$
$n \equiv_5 4$

then we just consider these possibilities.

$n \equiv_5 0 \Rightarrow n^2 \equiv_5 0$, or $5 | n^2$
$n \equiv_5 1 \Rightarrow n^2 \equiv_5 1$, or $5 | (n^2 - 1)$
$n \equiv_5 2 \Rightarrow n^2 \equiv_5 4$, or $5 | (n^2 - 4)$
$n \equiv_5 3 \Rightarrow n^2 \equiv_5 4$, or $5 | (n^2 - 4)$
$n \equiv_5 4 \Rightarrow n^2 \equiv_5 1$, or $5 | (n^2 - 1)$

So if we know $5 | p^2$ for some $p \in \mathbb{Z}$, the only possibility is that $5 | p$. $\square$

3. Determine whether the statements $Q \Rightarrow P$ and $P \vee \neg Q$ are logically equivalent.

P	Q	$Q \Rightarrow P$	$\sim Q$	$P \vee \sim Q$
T	T	T	F	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	T

Since $Q \Rightarrow P$ and $P \vee \sim Q$ have the same truth values for all possible circumstances, then they are logically equivalent. □

Great

4. $\sqrt{3}$ is irrational.

Well, suppose $\sqrt{3}$ is rational. We than can write $\sqrt{3} = \frac{p}{q}$ where p and q are co-prime integers and $q \neq 0$. Squaring both sides, $3 = \frac{p^2}{q^2}$, and multiplying q^2 on both sides, $3q^2 = p^2$. Due to q^2 being an integer, we can say 3 divides p^2 and thus 3 divides p. This means for some integer m, $3m = p$. We will plug this value for p in such that $3q^2 = (3m)^2$ and $3q^2 = 9m^2$. I will divide both sides by 3 so that $q^2 = 3m^2$. Due to m being an integer, we can say 3 divides q^2 and thus 3 divides q. We have just shown that p and q have a common factor of three, which is a contradiction to the assumption the variables are co-prime. Thus, by contradiction, $\sqrt{3}$ is irrational.

Nice!

5. For any $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ By induction

Prove Base case where $n=1$

$$\sum_{i=1}^{1} (1) = \frac{1(1+1)}{2} = \frac{(1)(2)}{2} = 1$$

The statement holds true for $n=1$. Assume this is true for ~~all~~ sort of. $n=k$, such that:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Now, it also needs to hold for $n=k+1$, so we will add $k+1$ to both sides

$$\sum_{i=1}^{k} i + k+1 = \frac{k(k+1)}{2} + k+1$$

[Common denominators

$$\sum_{i=1}^{k+1} i = \frac{k^2+k}{2} + \frac{2k+2}{2}$$

[add together

$$\sum_{i=1}^{k+1} i = \frac{k^2+3k+2}{2}$$

[factor

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

[manipulate

$$\sum_{i=1}^{k+1} i = \frac{k+1((k+1)+1)}{2}$$

This takes the same form $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n=k+1$ thus true!

By induction, the statement holds true for any $n \in \mathbb{Z}^+$ Okay!