**Examlet 2a** **Foundations of Advanced Math** **3/10/21**

1. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Express each as simply as possible:

   (a) $A \cup B$

   $A \cup B = \{1,2,3\}$

   (b) $A \cap B$

   $A \cap B = \{2\}$

   (c) $A - B$

   $A - B = \{1\}$

   Correct

   (d) $\mathcal{P}(A)$

   $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

   (e) $A \times B$

   $A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$

2. Biff says that each of the unions below is equal to $\mathbb{R}$. For each, either briefly support or refute his assertion.

(a) $\bigcup_{a\in\mathbb{Z}}(a,a+1)$

2 False, no integers are included

(b) $\bigcup_{a\in\mathbb{Z}}[a,a+1)$

2 True, any $x \in \mathbb{R}$ is either an integer or less than one greater than an integer

(c) $\bigcup_{a\in\mathbb{Z}}\{a,a+1\}$

2 False, this only contains integers

(d) $\bigcup_{a\in\mathbb{R}}\{a,a+1\}$

2 True, all elements of $\mathbb{R}$ are included by def, and all a+1 will still be in $\mathbb{R}$

(e) $\bigcup_{a\in\mathbb{Z}}(a,a+3)$

2 True, all integers are included via "a+1" inside (a, a+3), and all other reals are less than one above some "a+1"

Yep.

3.

$$A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i)$$

Proof:

Well, take $x \in A \cup \bigcap_{i \in I} B_i$, so that by definition, $x \in A \vee \forall i \in I, x \in B_i$. This can be rewritten as $\forall i \in I, x \in A \vee x \in B_i$, which by definition, is same as saying $x \in \bigcap_{i \in I} (A \cup B_i)$. Therefore, $A \cup \bigcap_{i \in I} B_i \subseteq \bigcap_{i \in I} (A \cup B_i)$.

Now, take $x \in \bigcap_{i \in I} (A \cup B_i)$, so that by definition, $\forall i \in I, x \in A \vee x \in B_i$ which can be rewritten as $x \in A \vee \forall i \in I, x \in B_i$, which means $x \in A \cup \bigcap_{i \in I} B_i$. Therefore, $\bigcap_{i \in I} (A \cup B_i) \subseteq A \cup \bigcap_{i \in I} B_i$ & thus

$$A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i) \; \blacksquare$$

Great

4. Show that if $a, b, c \in \mathbb{R}$ with $a < b$ and $c < 0$, then $ac > bc$. Give explicit justifications for each of your steps.

Proof: Well, let's take $c < 0$ and add $-c$ to both sides. So that $c - c < 0 - c \Rightarrow 0 < -c$ by the Comparison Addition Principle. So take $a < b$ and mutiply both sides by $-c$ for $a(-c) < b(-c) \Rightarrow -ac < -bc$ by the Comparison Mutiplication Principle. Then, we can add $ac$ and $bc$ to both sides. $ac - ac + bc < bc - bc + ac \Rightarrow bc < ac$ by the Comparison Addition Principle. Since $bc < ac$ is the same as $ac > bc$, then it is true for $a < b$ and $c < 0$ for $a, b, c \in \mathbb{R}$. $\square$

Wonderful!

5. $\forall x, y, z \in \mathbb{R}$, $|x + y + z| \leq |x| + |y| + |z|$.

Proof:

We know by Lemma 1 from exescise 2.6 that $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$ and $-|z| \leq z \leq |z|$. Adding these togethes, we have $-|x|-|y|-|z| \leq x+y+z \leq |x|+|y|+|z| \Rightarrow -(|x|+|y|+|z|) \leq x+y+z \leq |x|+|y|+|z|$. Now, if we apply Lemma 2 from exescise 2.6 (which states that ), we get

$$|x+y+z| \leq |x|+|y|+|z| \quad \square$$

Great