Examlet 2b **Foundations of Advanced Math** **3/10/21**

1. Let $A = \{1,2\}$ and $B = \{2,3\}$. Express each as simply as possible:

(a) $B \cup A$

$\{1,2,3\}$

(b) $B \cap A$

$\{2\}$

(c) $B - A$

$\{3\}$

Great

(d) $\mathcal{P}(B)$ $\rightarrow |P(B)| = 2^{|B|} = 4$

$\{\emptyset, \{2\}, \{3\}, \{2,3\}\}$

(e) $B \times A$

$\{(2,1),(2,2),(3,1),(3,2)\}$

2. Biff says that each of the unions below is equal to $\mathbb{R}$. For each, either briefly support or refute his assertion.

(a) $\bigcup_{a\in\mathbb{Z}}(a, a+1)$

This one is close to $\mathbb{R}$ but not exactly. It is $\mathbb{R}-\mathbb{Z}$, meaning it is every value in $\mathbb{R}$ that is not an integer because the interval doesn't include a or a+1.

(b) $\bigcup_{a\in\mathbb{Z}}(a, a+1]$

This one is $\mathbb{R}$ because it includes the value a+1 and ~~also~~ includes all values between a and a+1 including integers and all that is between them. Good Job, Biff :)

Nice

(c) $\bigcup_{a\in\mathbb{Z}}\{a, a+1\}$

This one is only $\mathbb{Z}$ as the union is of sets and not intervals. This would only have a and a+1 with nothing between them.

(d) $\bigcup_{a\in\mathbb{R}}\{a, a+1\}$

Yep. This is a union across all reals of a set which contains that real. So the union will include all reals. Good

(e) $\bigcup_{a\in\mathbb{N}}[a, a+3]$

Nope, $\mathbb{N}$ starts at 0, so this can't include numbers < 0 which are included in $\mathbb{R}$.

Good

3.

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

Well, let $x \in A \cap \bigcup_{i \in I} B_i$ so that $x \in A$ and $x \in \bigcup_{i \in I} B_i$. Then, $x \in B_i$ for some $i \in I$ and $x \in A$ so $x \in \bigcup_{i \in I} (A \cap B_i)$ and

$$A \cap \bigcup_{i \in I} B_i \subseteq \bigcup_{i \in I} (A \cap B_i).$$

Now, let $x \in \bigcup_{i \in I} (A \cap B_i)$ such that $\exists_i \in I$ where $x \in A$ and $x \in B_i$. This can also be written as $x \in A \cap \bigcup_{i \in I} B_i$, which means

$$\bigcup_{i \in I} (A \cap B_i) \subseteq A \cap \bigcup_{i \in I} B_i.$$

Nice!

Then, by mutual inclusion,

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

4. Show that if $a, b, c \in \mathbb{R}$ with $a < b$ and $c < 0$, then $ac > bc$. Give explicit justifications for each of your steps.

Proof: Well, let's take $c<0$ and add $-c$ to both sides. So that $c-c<0-c \Rightarrow 0<-c$ by the Comparison Addition Principle. So take $a<b$ and mutiply both sides by $-c$ for $a(-c)<b(-c) \Rightarrow -ac<-bc$ by the Comparison Mutiplication Principle. Then, we can add ac and bc to both sides. $ac-ac+bc<bc-bc+ac \Rightarrow bc<ac$ by the Comparison Addition Principle. Since $bc<ac$ is the same as $ac>bc$, then it is true for $a<b$ and $c<0$ for $a,b,c \in \mathbb{R}$. □

Wonderful!

5. $\forall x, y, z \in \mathbb{R}, |x + y + z| \leq |x| + |y| + |z|$.

Well, in knowing the Triangle Inequality Theorem, we can take elements x, y, and z and apply them to Lemma 1 from 2.6. So

$$-|x| \leq x \leq |x|;$$
$$-|y| \leq y \leq |y|; \text{ and}$$
$$-|z| \leq z \leq |z|.$$

Adding all the inequalities together via CAP we get

$$-(|x| + |y| + |z|) \leq x + y + z \leq |x| + |y| + |z|.$$

Finally, applying this to Lemma 2 on 2.6, we arrive at

$$|x + y + z| \leq |x| + |y| + |z|. \square$$

Great