Examlet 2 Advanced Geometry 3/17/21

1. a) State the definition of a *scalene triangle*.

A scalene triangle is a triangle with 3 sides of 3 different lengths.

b) State the definition of σ(△ABC).

The definition of $\sigma(\triangle ABC)$ is the angle sum of the interior angles of a triangle which is determined by...

$$\sigma(\triangle ABC) = \mu(\angle ABC) + \mu(\angle BCA) + \mu(\angle CAB)$$

c) State the Saccheri-Legendre Theorem.

For any triangle $\triangle ABC$,

$$\sigma(\triangle ABC) \leq 180^\circ$$

d) State the Alternate Interior Angles Theorem

~~tet bond~~. If ℓ and ℓ' are cut by a transversal t such that the pair of alternate interior angles are congruent, then ℓ and ℓ' are parallel

e) State the definition of *defect*.

the defect of a triangle is determined by

$$\delta(\triangle ABC) = 180^\circ - \sigma(\triangle ABC)$$

Great

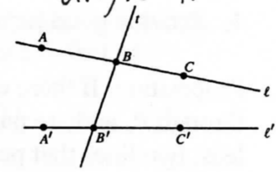
where $\triangle ABC$ is any triangle

2. Which of the following are equivalent (given the other postulates of neutral geometry) to the Euclidean Parallel Postulate? Check all that apply.

- ☐ The double perpendicular construction
- ☐ The Saccheri-Legendre Theorem
- ☑ Existence of rectangles
- ☑ Euclid's Postulate V
- ☑ Converse of the Alternate Interior Angles Theorem
- ☑ If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180°$.
- ☑ Clairaut's Axiom
- ☑ Every triangle has defect 0°.
- ☑ There exists a triangle whose defect is 0°.
- ☐ The Universal Hyperbolic Theorem
- ☐ a unicorn ate my petunias

angle theorem!!!

3. Provide good justifications in the blanks below for the corresponding statements:



Proposition: If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'.

Statement:	Reason:
Let ℓ and ℓ' be two lines cut by transversal t such that a pair of alternate interior angles is congruent.	this is our hypothesis
Choose points A, B, C, and A', B', C' as in the figure above. Suppose $\angle A'B'B \cong \angle B'BC$.	true by the ruler/point construction postulate & hypothesis
We must prove that ℓ is parallel to ℓ'. Suppose there exists a point D such that D lies on both ℓ and ℓ'.	this is the RAA hypothesis
If D lies on the same side of t as C, then $\angle A'B'B$ is an exterior angle for $\triangle BB'D$,	true by the definition of an exterior angle
while $\angle B'BC$ is a remote interior angle for $\triangle BB'D$.	true by the definition of a remote interior angle
This is a contradiction.	caused by the exterior angle theorem
In case D lies on the same side of t as A, then $\angle B'BC$ is an exterior angle and $\angle A'B'B$ is a remote interior angle for $\triangle BB'D$,	true by the definitions of exterior angle and of remote interior angle
and again we have a contradiction.	caused by the exterior angle theorem
Since D must lie on one of the two sides of t,	true by the plane separation postulate
we are forced to conclude that the proposition holds.	caused by the contradiction and definition of parallel lines

4. Provide good justifications in the blanks below for the corresponding statements:

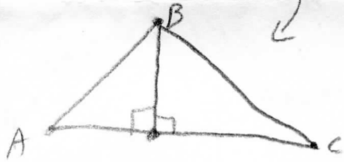
Proposition: If there exists one line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0, then for every line ℓ and for every external point P there exist at least two lines that pass through P and are parallel to ℓ.

Statement:	Reason:
S'pose there exists a line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0.	Hypothesis
Then the Euclidean Parallel Postulate fails.	by Euclidean Parallel Postulate terms: conditions
No rectangle exists.	Since Clairaut's Axiom is equivalent it too fails
Let ℓ be a line and P an external point.	by hypothesis/given
We must prove that there are at least two lines through P that are both parallel to ℓ. Drop a perpendicular to ℓ through P and call the foot of that perpendicular Q.	by hypothesis and existence of/uniqueness of perpendiculars
Let m be the line through P that is perpendicular to $\overleftrightarrow{PQ}$.	by existence of/uniqueness of perpendiculars
Choose a point R on ℓ that is different from Q and let t be the line through R that is perpendicular to ℓ.	by existence of/uniqueness of perpendiculars
Drop a perpendicular from P to t and call the foot of the perpendicular S.	by existence of/uniqueness of perpendiculars
Now $\square PQRS$ is a Lambert quadrilateral.	by Lambert's quadrilateral defn
But $\square PQRS$ is not a rectangle (reason?), so $\angle QPS$ is not a right angle and $\overleftrightarrow{PS} \neq m$.	by previous since Euclidean Parallel Postulate fails, Clairaut's Axiom fails, no rectangles exist
Nevertheless $\overleftrightarrow{PS}$ is parallel to ℓ,	by alternate interior angles theorem
so our proof is complete.	Because our proof is complete.

5. Explain, as if to someone intelligent but without any math background beyond high school, why if there exists one triangle whose defect is 0°, then every triangle has a defect of 0°.

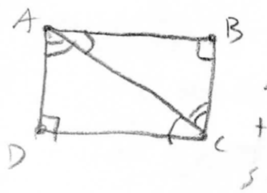
If we remove the assumption that parallel lines are unique, then the sum of angles in a triangle does not have to be 180, it could be less. The defect of a triangle is how much less than 180° the angle sum is.

If the defect of a triangle is 0° then its angle sum is 180°. Given such a triangle it must be able to be split into two right triangles like so:

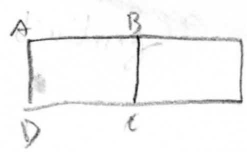


and those two triangles must have 180° sums since the sums of the angles of two triangles which make up a third equals the third's sum plus 180°.

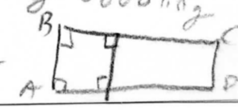
Take any of these 180° right triangles and mirror it over it's hypotenuse to form a rectangle.



See the figure that because the sum of ABCD's angles is equal to twice the right triangle, it must be a rectangle of angle sum 360°. These rectangles can be added or subtracted from eachother by doubling two

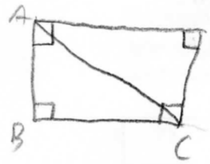


or cutting a perpendicular



In this fashion you can make any size rectangle.

Since any size rectangle can exist any right triangle



can be encased in a rectangle like so and the sum of the angles of the triangles must equal the rectangle. Since no triangle can have an angle sum greater than 180, they both need to be 180 to add up to 360, and therefore all right triangles have a sum of 180.

Finally, since any triangle can be split into two right triangles, both which must have angle sum 180°, they must also have 180° sums as 180+180=X+180

where X is the arbitrary triangle's angle sum.

therefore, if any triangle has angle sum 180°, all do.

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Nice Job!

You don't give yourself enough credit for ability to explain.