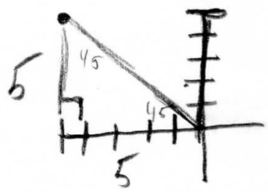
Exam 4 **Calculus 2** **4/21/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Convert the point with rectangular coordinates $(-5, 5)$ to polar coordinates (r, θ).

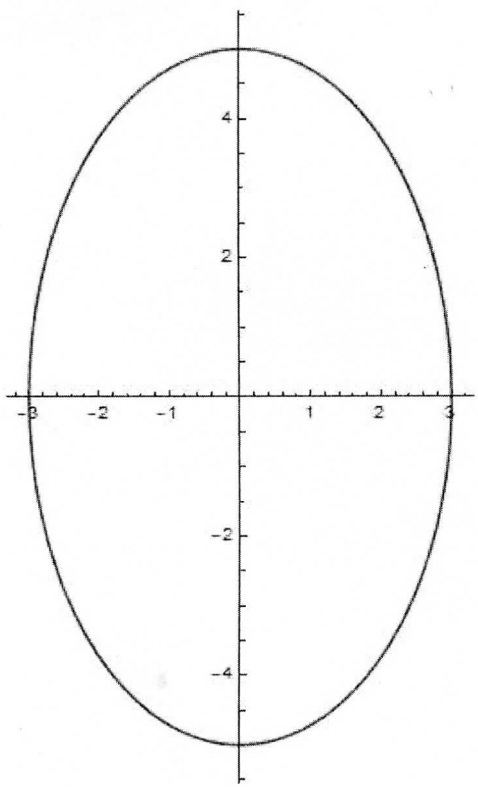


$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

$\left(\sqrt{50}, \frac{3\pi}{4}\right)$

Great!

2. Find an equation for the ellipse shown:



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$

3. Consider the curve defined by the parametric equations $x(t) = t^3 - 5t$ and $y(t) = 8t^2$. Set up an integral for the length of the loop of this curve.

Find t-values

$0 = t^3 - 5t = t(t^2-5) \quad t = \pm\sqrt{5}, \; t = 0$

$$L = \int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{(3t^2-5)^2 + (16t)^2}\, dt$$

Good

4. Set up an integral for the area of the region inside the curve with polar equation $r = 6\sin(5\theta)$.

$$A = 5\int_0^{\frac{\pi}{5}} \frac{1}{2}\left[6\sin(5\theta)\right]^2 d\theta$$

$$A = \frac{5}{2}\int_0^{\frac{\pi}{5}} 36\sin^2(5\theta)\, d\theta$$

Good

$0 = 6\sin(5\theta)$

$\sin(5\theta) = 0$

$\theta = \frac{1}{5}\sin^{-1}(0)$

5. Identify the graph of $y^2 - x^2 - 10y + 4x - 15 = 0$ as a parabola, hyperbola, or ellipse, give coordinates of its vertices, and sketch a decent graph.

don't forget -4 $+25$

$(y^2 - 10y + 25) - (x^2 - 4x + 4) = 15$ $40 - 4 = 36$

$-10 \cdot \frac{1}{2} = -5$

$-5^2 = 25$

$-4 \cdot \frac{1}{2} = -2$

$-2^2 = 4$

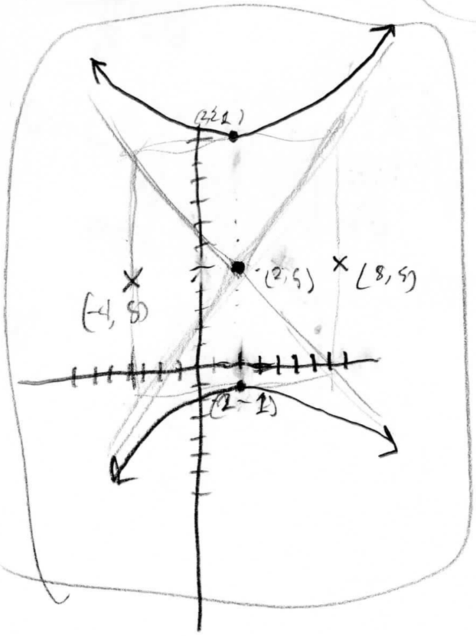
$$\frac{(y-5)^2 - (x-2)^2 = 36}{36}$$

$$\boxed{\frac{(y-5)^2}{6^2} - \frac{(x-2)^2}{6^2} = 1}$$

Center: (2, 5)

hyperbola

vertices: (2, 11) and (2, -1)



(2, 11) (2, -1)

6 wide
6 tall

$(y-5)^2$ 2

11 and -1

(2, 11) (2, -1)

Great

$(y-5)^2 = 6^2$

$y = 11$

$y = -1$

6. Write an integral for the area of the region inside the inner loop of $r = 1 + 2\cos(\theta)$.

Inner loop starts and ends where $r=0$, so:

$0 = 1 + 2\cos\theta$

$\frac{-1}{2} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{-1}{2}\right)$

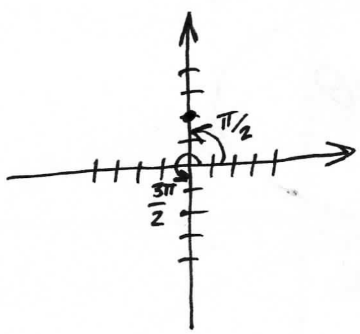
$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ From the graph I can see this order gets the inner loop.

$$\text{Area} = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2\, d\theta$$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. I am just totally confused. So like, I know with normal stuff, like, negative x is like the left half, right? And negative y is the bottom half, right? But I asked what half is negative with, like, this new r thingy, right? And the professor just looked at me funny, in front of like 300 people in the lecture, right? So I pretty much died and he just went on. I think I better drop."

Help Bunny out by explaining where points with negative r-values can be located.

So Bunny, with polar coordinates anyplace can have a negative r value. So think about, for example, a point two units up from the origin. The normal way to get



there is by facing direction $\theta = \frac{\pi}{2}$, so up the y-axis, and going two units in that direction. But another way to get to that exact same spot is to face direction $\theta = \frac{3\pi}{2}$, so down the negative y-axis, but then go two steps backwards. So really anyplace you can get to one way, you can also get to with an extra half-turn and the negative of that r-value. Anyplace can be reached with a negative-r.

8. Find the exact coordinates of all points on the graph of the curve with parametric equations $x(t) = t^3 - 6t, y(t) = t^2 - 5$ where the tangent line is vertical.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Vertical tangent line means $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = 3t^2 - 6 = 0$ $\quad t = \pm\sqrt{2}$

$x(-\sqrt{2}) = 4\sqrt{2}$ $\quad x(\sqrt{2}) = -4\sqrt{2}$

$y(-\sqrt{2}) = -3$ $\quad y(\sqrt{2}) = -3$

The tangent line is vertical at the points $(4\sqrt{2}, -3)$ and $(-4\sqrt{2}, -3)$

Excellent!

9. Find the exact (x, y) coordinates of all point(s) with horizontal tangent lines on the cardioid with polar equation $r = 1 + \cos\theta$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(r\sin\theta\right)}{\frac{d}{dt}\left(r\cos\theta\right)} = \frac{\frac{d}{dt}\left(\left(1+\cos\theta\right)\sin\theta\right)}{\frac{d}{dt}\left(\left(1+\cos\theta\right)\cos\theta\right)}$$

$$\text{So } \frac{dy}{dx} = \frac{-\sin\theta\sin\theta + \left(1+\cos\theta\right)\cos\theta}{-\sin\theta\cos\theta + \left(1+\cos\theta\right)\cdot -\sin\theta}$$

A line is horizontal when the slope's numerator is 0, so

$$-\sin^2\theta + \cos\theta + \cos^2\theta = 0$$

$$-\left(1-\cos^2\theta\right) + \cos\theta + \cos^2\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0 \qquad \text{Let } \alpha = \cos\theta$$

$$2\alpha^2 + \alpha - 1 = 0$$

$$(2\alpha - 1)(\alpha + 1) = 0$$

$$\text{So } \alpha = \tfrac{1}{2} \text{ or } \alpha = -1$$

$$\text{Or } \cos\theta = \tfrac{1}{2} \text{ or } \cos\theta = -1$$

$$\text{This means } \theta = \tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3} \text{ or } \theta = +\pi$$

For (x, y) coordinates:

$$\theta = \tfrac{\pi}{3} \rightarrow \quad x = \left(1+\cos\tfrac{\pi}{3}\right)\cos\tfrac{\pi}{3} = \tfrac{3}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4}$$
$$y = \left(1+\cos\tfrac{\pi}{3}\right)\sin\tfrac{\pi}{3} = \tfrac{3}{2}\cdot\tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{3}}{4}$$

$$\theta = \tfrac{5\pi}{3} \rightarrow \quad x = \tfrac{3}{4},\ y = \tfrac{-3\sqrt{3}}{4}$$

$\theta = \pi \rightarrow \quad x = 0,\ y = 0$ (this one only kinda counts - it's a cusp so tangent lines are weird.)

10. Find the area enclosed by the loop of the curve with parametric equations $x(t) = t^3 - 3t$, $y(t) = t^2 + t + 1$

From the graph, it looks like the crossing point is $(-2, 3)$, so:

$t^3 - 3t = -2$ or $t^2 + t + 1 = 3$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$t = -2$ or $t = 1$

Hey! These satisfy the other equation too!

$$\text{So Area} = \int_{-2}^{1} \underbrace{(t^2+t+1)}_{y(t)} \underbrace{(3t^2-3)}_{x'(t)}\, dt$$

$$= \int_{-2}^{1} (3t^4 + 3t^3 + 3t^2 - 3t^2 - 3t - 3)\, dt$$

$$= \int_{-2}^{1} (3t^4 + 3t^3 - 3t - 3)\, dt$$

$$= \left[\frac{3}{5}t^5 + \frac{3}{4}t^4 - \frac{3}{2}t^2 - 3t \right]_{-2}^{1}$$

$$= \left(\frac{3}{5} + \frac{3}{4} - \frac{3}{2} - 3 \right) - \left(\frac{-96}{5} + 12 - 6 + 6 \right)$$

$$= \frac{81}{20}$$