Examlet 1 **Foundations of Advanced Math** **2/4/22**

1. The sum of any two throddodd integers is throdd.

n throddodd mean that $n=3a+2$ for some $a\in\mathbb{Z}$.
m throddodd mean that $m=3b+2$ for some $b\in\mathbb{Z}$.

$$
\begin{aligned}
n+m &= (3a+2)+(3b+2) \\
&= 3a+3b+4 \\
&= 3a+3b+3+1 \\
&= 3(a+b+1)+1
\end{aligned}
$$

$(a+b+1)$ is an integer by closure. So, $n+m = 3(\text{int.})+1$
Therefore, the sum of any two throddodd integers is throdd by definition. $\square$

Great!

2. Show that if $p \in \mathbb{Z}$ and $p^2 \equiv_5 0$ then $p \equiv_5 0$.

Well, it's straightforward to show that if $p\equiv_5 1$ then $p^2\equiv_5 1$, if $p\equiv_5 2$ then $p^2\equiv_5 4$, if $p\equiv_5 3$ then $p^2\equiv_5 4$, and if $p\equiv_5 4$ then $p^2\equiv_5 1$. Then the only remaining possibility is $p\equiv_5 0$, for which $p^2\equiv_5 0$, so it's only if $p\equiv_5 0$ that $p^2\equiv_5 0$, i.e. $p^2\equiv_5 0$ for $p\in\mathbb{Z} \Rightarrow p\equiv_5 0$. $\square$

3. The statements $P \Rightarrow (Q \wedge R)$ and $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ are logically equivalent.

We construct the following truth table:

P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$ ✯	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$ ✯
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the two columns marked with a ✯ match, meaning that they have the same truth values under all circumstances, then the statements $P \Rightarrow (Q \wedge R)$ and $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ are logically equivalent. □

Excellent!

4. $\sqrt{5}$ is irrational.

Proof: let's suppose $\sqrt{5}$ is rational. So, we can say $\sqrt{5} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$ and the fraction has been reduced so p and q have no common factors.
We can square both sides of the equation to get rid of the square root.

$$\sqrt{5}^2 = \left(\frac{p}{q}\right)^2 \Rightarrow 5 = \frac{p^2}{q^2}$$

multiply both sides by q^2:

$$5q^2 = p^2$$

p^2 is then divisable by 5 so, we know p is also divisable by 5.
So p can be written as $p = 5r$ where $r \in \mathbb{Z}$.
Subsitute $p = 5r$:

$$5q^2 = (5r)^2 \quad \text{distribute the square}$$

$$5q^2 = 25r^2. \quad \text{reduce the equation}$$

$$q^2 = 5r^2.$$

This means that q^2 is also divisable by 5 and so q is divisable 5. This contradicts our supposition that p and q are co-prime, leaving us to conclude $\sqrt{5}$ is irrational. □

Good

5. For any $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n}(2i-1) = n^2$$

proof: Base case $n=1$. $\sum_{i=1}^{1}(2i-1) = 1$

$1^2 = 1$ where is true.

Assume that it is also true for $n=k$ such that

$$\sum_{i=1}^{k}(2i-1) = k^2.$$

Now, when $n=k+1$.

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + [2(k+1)-1]$$

$$= k^2 + 2k + 1 = (k+1)^2$$

which match the form of $\sum_{i=1}^{n}(2i-1) = n^2$ when $n=k+1$

Therefor, the statment is true $\triangleright$ Nice.