1. Let $f, g : \mathbb{R} \to \mathbb{R}$.

   (a) If f and g are both increasing, then $f + g$ is increasing.

proof: f and g are both increasing means that

let $x < y$. $f(x) \leq f(y)$. and $g(x) \leq g(y)$.

take $(f+g)(x) = f(x) + g(x) \leq f(y) + g(x) \leq f(y) + g(y)$

where $f(y) + g(y)$ can be written as $f+g(y)$.

$\therefore f+g(x) \leq f+g(y)$ so $f+g$ is increasing

Great

   (b) If $f + g$ is increasing, then f and g are both increasing.

counter example

let $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 3x$. which is an increasing function.

$g : \mathbb{R} \to \mathbb{R}$ $g(x) = -x$ which is decreasing.

$f+g(x) = f(x) + g(x)$

$= 3x - x = 2x$ where is increasing function

we can see for $f+g$ is increasing.

f or g are not nessery to both be increasing function.

2. If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is injective.

$f: A \to B$ being injective means $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.
$g: B \to C$ being injective means that $g(b_1) = g(b_2) \Rightarrow b_1 = b_2$

Suppose we had that $(g \circ f)(a_1) = (g \circ f)(a_2)$, so $g(f(a_1)) = g(f(a_2))$. Well, g is injective so we know that $f(a_1) = f(a_2)$. f is also injective, so we would get that $a_1 = a_2$. Therefore, we have that $(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow a_1 = a_2$, so $g \circ f$ is injective by definition. □

Excellent!

$g: B \to A$
$f(g(b)) = b$
$g(f(a)) = a$

3. If $f : A \to B$ is a bijection, then f is invertible.

Well, since $f: A \to B$ is bijective, it must be surjective so that $\forall b \in B$, $\exists\, a \in A$ such that $f(a) = b$. If we have a function $g: B \to A$ defined by $g(b) = a$, then $g(b) \in A$ so an image will be created for all $a \in A$. because there exists an $a \in A$ for all $b \in B$. We must also ensure that a single b does not get sent to more than a single $a \in A$ (injective). If we have that a $b \in B$ is sent to two different $a_1, a_2 \in A$, this would mean that $a_1 = g(b) = a_2$. This would also mean that $f(a_1) = b = f(a_2)$, but since f is injective, we know that $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. by definition of inverse functions

Therefore, we have a function $g: B \to A$ such that $f(g(b)) = b$ and $g(f(a)) = a$, so $f: A \to B$ is invertible. □

Okay.

4. If A is equipollent to B, and B is equipollent to C, then A is equipollent to C.

If A is equipollent to B, this means there exists a bijection $f: A \to B$.

And B is equipollent to C, this means there exists a bijection $g: B \to C$.

Now, lets consider $g \circ f: A \to C$. We know that the composition of f and g, two bijective functions, results in $g \circ f$ being bijective from previous work. Thus, there exists a bijection $g \circ f: A \to C$. Therefore, by def of equipollent, A is equipollent to C. $\square$

Excellent!

5. The set of integers is countable.

x	$f(x)$
-2	4
-1	2
0	0
1	1
2	3

We must define a bijection $f: \mathbb{Z} \to \mathbb{N}$.

We can define $f(x) = \begin{cases} 2x-1 & x > 0 \\ 2|x| & \text{if } x \leq 0 \end{cases}$

This maps positive integers to odd natural numbers, negative integers to even naturals, and 0 to itself. This must be a bijection to establish equipollence. We can tell that f is injective as each element of the codomain, $\mathbb{N}$, can only be produced by a single element of the domain, $\mathbb{Z}$. Additionally, each element of $\mathbb{N}$ can be produced by an element of $\mathbb{Z}$ such that $f(z) = n$. Thus, $\mathbb{Z}$ is equipollent to $\mathbb{N}$ and $\mathbb{Z}$ is countable. $\square$

Excellent!