Exam 3 **Calc 1** **3/24/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Suppose that $f(x) = (\log_5 x)(2^x)$. Find $f'(x)$. product rule

$$\left(\frac{1}{\ln 5 \cdot x} \cdot 2^x \right) + \left(\log_5 x \cdot (\ln 2 \cdot 2^x) \right)$$

Good

2. Let $f(x) = 7x\sin^{-1}(x)$.

(a) Find $f'(x)$. product rule

$f'g + fg'$

$$\left(7\sin^{-1}(x)\right) + \left(7x \cdot \frac{1}{\sqrt{1-x^2}}\right)$$

Great!

(b) Find $f'(0.6)$.

$$7\sin^{-1}(.6) + \left(7(.6) \cdot \left(\frac{1}{\sqrt{1-(.6)^2}}\right)\right)$$

$$4.504 + (4.2 \cdot 1.25)$$

$$f'(.6) = 9.754$$

Yes

3. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	6	3	5	1	4	2
$g(x)$	2	1	6	3	5	4
$f'(x)$	2	4	1	5	7	8
$g'(x)$	5	9	7	11	2	12

(a) If $h(x) = \tan^{-1}(f(x))$, what is $h'(2)$ and why? Chain rule

$h(2) = \tan^{-1} f(2)$

$\frac{1}{1+f(2)^2} \cdot f'(2)$ $\qquad \frac{1}{1+3^2} \cdot 4 =$

$\frac{1}{10} \cdot 4 = .4$ Good

(b) If $h(x) = f(x) \cdot \ln x$, what is $h'(5)$ and why? Product rule

$h(5) = f(5) \cdot \ln 5$

$f(5)' \cdot \ln 5 + f(5) \cdot \frac{1}{5}$

$7 \cdot \ln 5 + 4 \cdot \frac{1}{5} = 12.06$ Good

(c) If $h(x) = e^{g(x)}$, what is $h'(4)$ and why? Chain Rule

$h(4) = e^{g(4)}$

$e^{g(4)} \cdot g'(4)$

$e^3 \cdot 11 = 220.9$ Yes

4. Why is the derivative of $\ln x$ equal to $\frac{1}{x}$?

We know $e^{\ln x} = x$

differentiate: $e^{\ln x} \cdot (\ln x)' = 1$

$(\ln x)' = \frac{1}{e^{\ln x}}$

$(\ln x)' = \frac{1}{x}$

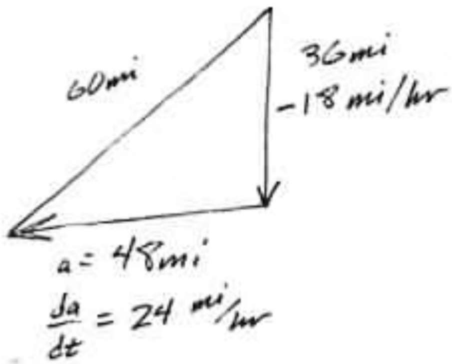
Good

$e^{\ln x} = x$

5. A train leaves Boston heading west at noon travelling 24 mi/hr. A second train is heading toward Boston from the north at 18 mi/hr and will arrive at 4pm. How fast is the distance between the trains changing at 2pm?

I. Find the distance @ 2pm:

$$a^2 + b^2 = c^2$$



$$36^2 + 48^2 = c^2$$

$$3600 = c^2$$

$$c = 60$$

II. Find the change in distance @ 2pm:

Differentiate: $2a\,\frac{da}{dt} + 2b\,\frac{db}{dt} = 2c\,\frac{dc}{dt}$

$$2(48)(24) + 2(36)(-18) = 2(60)\,\frac{dc}{dt}$$

$$1008 = 120\,\frac{dc}{dt}$$

$$\frac{42}{5} = \frac{dc}{dt}$$

$$\frac{dc}{dt} = 8.4 \text{ mi/hr}$$

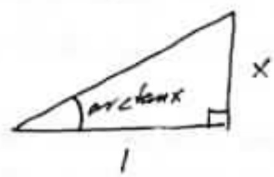
6. Why is the derivative of $\arctan x$ equal to $\frac{1}{1+x^2}$?

I know: $\tan(\arctan x) = x$ because they're inverses

Differentiating: $\sec^2(\arctan x) \cdot (\arctan x)' = 1$

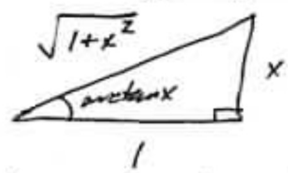
Solve: $(\arctan x)' = \dfrac{1}{\sec^2(\arctan x)}$

Now I'll build a right triangle with an angle whose tangent is x:



Find the hypotenuse:

$$a^2 + b^2 = c^2$$
$$(1)^2 + (x)^2 = c^2$$
$$c = \sqrt{1+x^2}$$



So the cosine of that angle is adjacent over hypotenuse,

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

And secant is the reciprocal:

$$\sec(\arctan x) = \sqrt{1+x^2}$$

So from above:

$$(\arctan x)' = \frac{1}{\sec^2(\arctan x)} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}. \ \square$$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude! I just broke calculus! There was this question where we had to do the derivadid of ln of $5x$, right? So I did it and when I simplified, like cancelled, I got 1 over x. So obviously that's not possible, because just ln of x has that for its derivadid, right? So there's no way two different functions have the same derivadid, right?"

Help Biff by explaining as clearly as you can why this isn't really surprising.

So Biff, I'm glad you realized to use the Chain Rule on

$$[\ln(5x)]' = \frac{1}{5x} \cdot 5 = \frac{1}{x}.$$

But as far as having the same derivative as $\ln x$, that's really not surprising. If you look at the graphs, $\ln(5x)$ and $\ln x$ look a lot alike, just vertically shifted. So it's like x^2 and x^2+5 both have $2x$ for their derivatives.

But then once you realize all that, it might trigger your memory that $\ln(5x) = \ln 5 + \ln x$, because of those properties of logarithms everybody mostly forgets. So that means these really are vertical shifts of each other, just like x^2 and x^2+5, so it makes complete sense that they have the same derivative.

8. Carbon dioxide emissions from China have been estimated at 10.30 gigatons in 2016 and 11.30 gigatons in 2020. [World Resources Institute data]

(a) Find a function of the form $f(x) = Ab^x$ for these emissions, where x is in years after 2016.

For 2016: $10.3 = A \cdot b^0$
$A = 10.3$

For 2020: $11.3 = 10.3 \cdot b^4$
$b = \left(\frac{11.3}{10.3}\right)^{1/4} \approx 1.023435$

(b) What does your formula project the carbon dioxide emissions will be in 2030?

$$f(x) \approx 10.3 \cdot (1.023435)^{14}$$
$$\approx 14.2456$$

(c) What does your formula project the rate of growth of carbon dioxide emissions will be in 2030?

$$f'(x) \approx 10.3 \cdot (\ln 1.023435) \cdot (1.023435)^t$$
$$f'(14) \approx 10.3 \cdot (\ln 1.023435) \cdot (1.023435)^{14}$$
$$\approx 0.3299957$$

9. Let $F(x) = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x$. Find $F'(x)$.

$$F'(x) = \frac{1}{2} \cdot \sqrt{1-x^2} + \frac{x}{2} \cdot \frac{1}{2}\left(1-x^2\right)^{-1/2} \cdot -2x + \frac{1}{2}\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\sqrt{1-x^2}}{2} + \frac{-x^2}{2\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}$$

$$= \frac{\sqrt{1-x^2}}{2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} + \frac{-x^2}{2\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}$$

$$= \frac{1-x^2}{2\sqrt{1-x^2}} + \frac{-x^2}{2\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}$$

$$= \frac{2\left(1-x^2\right)}{2\sqrt{1-x^2}}$$

$$= \frac{1-x^2}{\sqrt{1-x^2}}$$

$$= \sqrt{1-x^2}$$

10. A training program models participants' learning curves with the function

$$P(t) = \frac{2.9e^t}{5.0 + 0.22e^t}$$

where t is the number of training days and $P(t)$ is the participant's expected performance rating.

(a) Find $P^{-1}(x)$

$$y = \frac{2.9e^t}{5.0+0.22e^t}$$

$$y(5+0.22e^t) = 2.9e^t$$

$$5y + 0.22ye^t = 2.9e^t$$

$$5y = 2.9e^t - 0.22ye^t$$

$$5y = e^t(2.9 - 0.22y)$$

$$e^t = \frac{5y}{2.9-0.22y}$$

$$t = \ln\left(\frac{5y}{2.9-0.22y}\right)$$

$$P^{-1}(x) = \ln\frac{5x}{2.9-0.22x}$$

(b) Find the slope of the tangent line to $P^{-1}(x)$ at $x = 13$.

$$(P^{-1})'(x) = \frac{1}{\frac{5x}{2.9-0.22x}} \cdot \frac{5\cdot(2.9-0.22x) - 5x\cdot -0.22}{(2.9-0.22x)^2}$$

$$= \frac{\cancel{2.9-0.22x}}{5x} \cdot \frac{14.5 - 1.1x + 1.1x}{(2.9-0.22x)^{\cancel{2}}}$$

$$= \frac{14.5}{5x(2.9-0.22x)}$$

$$(P^{-1})'(13) \approx 5.5769$$