Exam 1a **Calculus 2** **2/2/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Evaluate $\int \sin^4\theta\cos\theta\,d\theta$

let $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta\,d\theta$

$= \int u^4\,du$

$= \frac{1}{5}u^5 + C$

Excellent!

$= \frac{1}{5}(\sin\theta)^5 + C$

2. Evaluate $\int xe^x\,dx$

$uv - \int v\,du$ IBP

$u = x$ $v = e^x$

$du = 1\,dx$ $dv = e^x\,dx$

$xe^x - \int e^x \cdot 1\,dx$

$= xe^x - \int e^x\,dx$

Great!

$= \boxed{xe^x - e^x + C}$

3. Write the appropriate form for a partial fractions decomposition of the function

$$\frac{2(x+1)}{(x-2)^2(x-1)^2(x^2+2)^2}$$

$$= \boxed{\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{Ex+F}{(x^2+2)} + \frac{Gx+H}{(x^2+2)^2}}$$

Great

4. Evaluate $\int_e^{e^5} \frac{dx}{x\sqrt{\ln x}}$

$$= \int_e^{e^5} \frac{1}{x\sqrt{\ln x}}\, dx$$

u sub

$u = \ln x$

$du = \frac{1}{x}\, dx$

$dx = x\, du$

$$= \int_{x=e}^{x=e^5} \frac{1}{\cancel{x}\, u^{1/2}}\, \cancel{x}\, du$$

$$= \int_{x=e}^{x=e^5} u^{-1/2}\, du$$

$$2u^{1/2} \Big|_e^{e^5}$$

$$= 2(\ln(x))^{1/2} \Big|_e^{e^5}$$

Good

$$= \boxed{2(\ln(e^5))^{1/2} - 2(\ln(e))^{1/2}}$$

5. Evaluate $\int_3^{\infty} e^{p/2}\,dp = \lim_{b\to\infty} \int_3^b e^{p/2}\,dp$

$$= \lim_{b\to\infty} \int_{p=3}^{p=b} e^u \cdot 2\,du$$

$$= \lim_{b\to\infty} 2 \cdot e^u \Big|_{p=3}^{p=b}$$

$$= \lim_{b\to\infty} 2 \cdot e^{p/2} \Big|_3^b$$

$$= \lim_{b\to\infty} 2e^{b/2} - 2e^{3/2}$$

diverges!

Let $u = \frac{p}{2}$

$\frac{du}{dp} = \frac{1}{2}$

$2\,du = dp$

6. Evaluate $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

let $u = 1 - x^2$

$du/dx = -2x$

$du/-2x = dx$

$u = 1 - x^2$

$u - 1 = -x^2$

$1 - u = x^2$

$$-\frac{1}{2}\int \frac{x^{\cancel{3}}}{\sqrt{u}} \cdot \frac{du}{\cancel{x}}$$

$$-\frac{1}{2}\int \frac{x^2}{\sqrt{u}}\,du$$

$$-\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du$$

$$-\frac{1}{2}\int u^{-1/2}(1-u)\,du$$

$$-\frac{1}{2}\int u^{-1/2} - u^{1/2}\,du$$

$$-\frac{1}{2}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right)$$

$$\boxed{-\frac{1}{2}\left(2(1-x^2)^{1/2} - \frac{2}{3}(1-x^2)^{3/2}\right) + C}$$

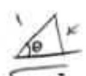
Good

7. Star is a calculus student at Enormous State University, and they're having some trouble. Star says "Yikes! Calc 2 is like a totally different thing than Calc 1. These problems are so long! And sometimes I wonder why they pick the things they do, like for a trig sub one, why do they do $x = \sin\theta$? Would it work if you did $x = \cos\theta$?"

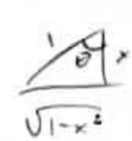
Help Star out. Explain to them as clearly as possible whether their alternative works well, and why.

using the last problem $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$, $\sin\theta$ for x works really well.

I chose to use $\sin\theta$ because it works well with the triangle.



$\sin\theta$ in this case is $\frac{x}{1}$, which becomes $x = \sin\theta$. Of course we can move around the θ in the triangle to try to use $\cos\theta$. If we set it up like this



$\cos\theta$ would be $\frac{x}{1}$ as well. Calc 2 is a very open class with several ways to do certain problems. So yes $x = \cos\theta$ would work as well. $\int \frac{\cos^3\theta}{\sqrt{1-\cos^2}}\,(-\sin\theta\,d\theta)$

$-\int \frac{\cos^3\theta}{\sqrt{\sin^2\theta}}\,(-\sin\theta)\,d\theta = -\int \cos^3\theta\,d\theta$. Then go from there.

With trig sub there are many options. Good

8. Derive Line 87 from the Table of Integrals,

$$\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$$

$u = \sin^{-1} u \qquad v = u$

$u' = \frac{1}{\sqrt{1-u^2}} \qquad v' = 1$

$= u\sin^{-1} u - \int \frac{u}{\sqrt{1-u^2}}\,du$

let $v = 1 - u^2$

$dv/du = -2u$

$dv/-2u = du$

$+^{+}2\int \frac{\cancel{u}}{\sqrt{v}} \cdot \frac{dv}{\cancel{u}}$

$2\int v^{-1/2}\,dv$

$\cancel{2}\left(\cancel{\tfrac{1}{2}} v^{1/2}\right) + C$

$\sqrt{v}$

$\boxed{u\sin^{-1} u + \sqrt{1-u^2} + C}$

Excellent!

9. Derive Line 30 from the Table of Integrals,

$$\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$$\int \sqrt{a^2 - u^2}\, du = \int \sqrt{a^2 - a^2 \sin^2\theta} \cdot a\cos\theta\, d\theta$$

Let $u = a\sin\theta$

$\frac{du}{d\theta} = a\cos\theta$

$du = a\cos\theta\, d\theta$

$$= \int a\sqrt{1 - \sin^2\theta} \cdot a\cos\theta\, d\theta$$

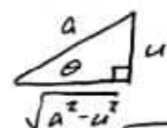
$$= a^2 \int \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$

Line 64

$\sin\theta = \frac{u}{a}$

$$= a^2\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$$



$$= a^2\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C$$

$\cos\theta = \frac{\sqrt{a^2 - u^2}}{a}$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2} \cdot \frac{u}{a} \cdot \frac{\sqrt{a^2 - u^2}}{a} + C$$

$$= \frac{a^2}{2}\sin^{-1}\theta + \frac{u}{2}\sqrt{a^2 - u^2} + C$$

10. Evaluate $\int \frac{1}{1+x^3}\,dx$ [Hint: $1 + x^3 = (1 + x)(1 - x + x^2)$]

I wish:

$$\frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}$$

$$1 = A(1-x+x^2) + (Bx+C)(1+x)$$

If $x=-1$:

$$1 = A(3) + 0$$
$$A = \frac{1}{3}$$

Matching x^2 coefficients:

$$0 = A + B$$
$$0 = (\tfrac{1}{3}) + B$$
$$B = \tfrac{-1}{3}$$

Matching Constant Terms:

$$1 = A + C$$
$$1 = (\tfrac{1}{3}) + C$$
$$C = \tfrac{2}{3}$$

$$x^2 - x + 1 = x^2 - x + \tfrac{1}{4} + 1 - \tfrac{1}{4}$$
$$= (x - \tfrac{1}{2})^2 + \tfrac{3}{4}$$

So
$$\int \frac{1}{1+x^3}\,dx = \int\left(\frac{1/3}{1+x} + \frac{-1/3\,x + 2/3}{1-x+x^2}\right)dx$$

$$= \frac{1}{3}\int \frac{1}{1+x}\,dx - \frac{1}{3}\int \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{(u+\frac{1}{2})-2}{u^2+3/4}\,du$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{u}{u^2+3/4}\,du + \frac{1}{3}\int \frac{+3/2}{u^2+3/4}\,du$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{\cancel{u}}{w}\cdot\frac{dw}{2\cancel{u}} + \frac{1}{3}\cdot\frac{+3}{2}\int \frac{1}{u^2+3/4}\,du$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|w| + \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\arctan\left(\frac{u}{\sqrt{3}/2}\right) + C$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|u^2+\tfrac{3}{4}| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Let $u = x - \frac{1}{2}$

$$\frac{du}{dx} = 1$$
$$du = dx$$

So $x = u + \frac{1}{2}$

Let $w = u^2 + 3/4$

$$\frac{dw}{du} = 2u$$
$$\frac{dw}{2u} = du$$