Exam 1b **Calculus 2** **2/2/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Evaluate $\int \sin^2\theta \cos\theta \, d\theta$

$u = \sin\Theta$

$\frac{du}{d\Theta} = \cos\Theta$

$du = \cos\Theta \, d\Theta$

$\int u^2 \, d\Theta$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}\sin^3\Theta + C$

Great!

2. Evaluate $\int xe^x \, dx$

$\int xe^x \, dx$

$u = x \quad v = e^x$

$du = 1 \quad v' = e^x$

$uv - \int v \, du$

$= xe^x - \int e^x$

$= xe^x - e^x + C$

Excellent!

3. Write the appropriate form for a partial fractions decomposition of the function

$$\frac{2(x^4+1)}{(x-2)^2(x-1)(x^2+2)^2}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{Dx+E}{(x^2+2)} + \frac{Fx+G}{(x^2+2)^2}$$

Good

4. Evaluate $\int_e^{e^5} \frac{dx}{x\sqrt{\ln x}}$

$$= \int_e^{e^5} \frac{1}{x\sqrt{\ln x}}\, dx$$

u sub

$u = \ln x$

$du = 1/x \; dx$

$dx = x\, du$

$$= \int_{x=e}^{x=e^5} \frac{1}{\cancel{x}\, u^{1/2}}\, \cancel{x}\, du$$

$$= \int_{x=e}^{x=e^5} u^{-1/2}\, du$$

$$2u^{1/2} \Big|_e^{e^5}$$

$$= 2(\ln(x))^{1/2} \Big|_e^{e^5}$$

Good

$$= \boxed{2(\ln(e^5))^{1/2} - 2(\ln(e))^{1/2}}$$

5. Evaluate $\int_3^{\infty} e^{-5p}\, dp$

$\lim_{b \to \infty} \int_3^b e^{-5p}\, dp$

let $u = -5p$

$\frac{du}{dp} = -5$

$= \frac{1}{-5} \cdot \lim_{b \to \infty} \int e^u\, du$

$= \frac{-1}{5} \cdot \lim_{b \to \infty} \left[e^u \right]$

$= \frac{-1}{5} \lim_{b \to \infty} \left[e^{-5p} \right]_3^b = \lim_{b \to \infty} \left[\frac{-1}{5} e^{-5b} + \frac{1}{5} e^{-15} \right] = \frac{-1}{5} \overset{0}{e^{-\infty}} + \frac{1}{5} e^{-15}$

$= 0 + \frac{1}{5} e^{-15} = \boxed{\frac{1}{5e^{15}}}$

Excellent!

6. Evaluate $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

let $u = 1-x^2$

$du/dx = -2x$

$du/-2x = dx$

$u = 1-x^2$

$u-1 = -x^2$

$1-u = x^2$

$$-\frac{1}{2}\int \frac{x^{\cancel{3}}}{\sqrt{u}} \cdot \frac{du}{\cancel{x}}$$

$$-\frac{1}{2}\int \frac{x^2}{\sqrt{u}}\,du$$

$$-\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du$$

$$-\frac{1}{2}\int u^{-1/2}(1-u)\,du$$

$$-\frac{1}{2}\int u^{-1/2} - u^{1/2}\,du$$

$$-\frac{1}{2}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right)$$

$$\boxed{-\frac{1}{2}\left(2(1-x^2)^{1/2} - \frac{2}{3}(1-x^2)^{3/2}\right) + C}$$

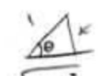
Good

7. Star is a calculus student at Enormous State University, and they're having some trouble. Star says "Yikes! Calc 2 is like a totally different thing than Calc 1. These problems are so long! And sometimes I wonder why they pick the things they do, like for a trig sub one, why do they do $x = \sin\theta$? Would it work if you did $x = \cos\theta$?"

Help Star out. Explain to them as clearly as possible whether their alternative works well, and why.

using the last problem $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$, $\sin\theta$ for x works really well.

I chose to use $\sin\theta$ because it works well with the triangle.



$\sin\theta$ in this case is $\frac{x}{1}$, which becomes $x = \sin\theta$. Of course we can move around the θ in the triangle to try to use $\cos\theta$. If we set it up like this

$\cos\theta$ would be $\frac{x}{1}$ as well. Calc 2 is a very open class with several ways to do certain problems. So yes $x = \cos\theta$ would work as well. $\int \frac{\cos^3\theta}{\sqrt{1-\cos^2}}(-\sin\theta\,d\theta)$

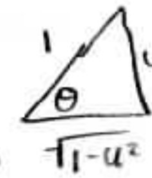
$-\int \frac{\cos^3\theta}{\sqrt{\sin^2\theta}}(-\sin\theta)\,d\theta = -\int \cos^3\theta\,d\theta$. Then go from there.

With trig sub there are many options. Good

$$\theta = \sin^{-1}(u)$$

8. Derive Line 87 from the Table of Integrals,

$$\int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1-u^2} + C$$



$= \int \sin^{-1}(\sin\theta) \cos\theta \, d\theta$ Let $u = \sin\theta$

$= \int \theta \cos\theta \, d\theta$ $\quad u = \theta \quad v = \sin\theta$, $\quad u' = d\theta \quad v' = \cos\theta$

$\dfrac{du}{d\theta} = \cos\theta$

$du = \cos\theta \, d\theta$

$= uv - \int u'v \, d\theta$

$= \theta\sin\theta - \int \sin\theta \, d\theta$

Clever!

$= \theta\sin\theta + \cos\theta \, d\theta + C$

$= \sin^{-1}(u) * u + \sqrt{1-u^2} + C$

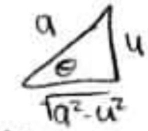
$= u\sin^{-1} u + \sqrt{1-u^2} + C$

$\sqrt{a^2-u^2} = a\sin\theta$

9. Derive Line 30 from the Table of Integrals,

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$\theta = \sin^{-1}\left(\frac{u}{a}\right)$



$= \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta\, d\theta$

Let $u = a\sin\theta$

$\frac{du}{d\theta} = a\cos\theta$

$du = a\cos\theta\, d\theta$

$= \int \sqrt{a^2(1-\sin^2\theta)} \times a\cos d\theta$

$= \int a\sqrt{\cos^2\theta} \times a\cos d\theta$

$= a^2 \int \cos^2\theta\, d\theta \longrightarrow$ line 64

$= a^2\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$

$= \frac{a^2}{2}\theta + \frac{1}{4}\underbrace{\sin 2\theta}_{=2\sin\theta\cos\theta} + C$

$\sin\theta = \frac{u}{a}$

$\cos\theta = \frac{\sqrt{a^2-u^2}}{a}$

$= \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + \frac{1}{4}(2)u\sqrt{a^2-u^2} + C$

$= \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + \frac{1}{2}u\sqrt{a^2-u^2} + C$

Nice!

$= \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

10. Evaluate $\int \frac{1}{1+x^3}\,dx$ [Hint: $1+x^3=(1+x)(1-x+x^2)$]

I wish:

$$\frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}$$

$$1 = A(1-x+x^2) + (Bx+C)(1+x)$$

If $x=-1$:

$$1 = A(3)+0$$

$$A = \tfrac{1}{3}$$

Matching x^2 coefficients:

$$0 = A + B$$

$$0 = (\tfrac{1}{3}) + B$$

$$B = \tfrac{-1}{3}$$

Matching Constant Terms:

$$1 = A + C$$

$$1 = (\tfrac{1}{3}) + C$$

$$C = \tfrac{2}{3}$$

$$x^2 - x + 1 = x^2 - x + \tfrac{1}{4} + 1 - \tfrac{1}{4}$$

$$= \left(x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$$

So $\int \frac{1}{1+x^3}\,dx = \int\left(\frac{1/3}{1+x} + \frac{-1/3\,x + 2/3}{1-x+x^2}\right)dx$

$$= \frac{1}{3}\int \frac{1}{1+x}\,dx - \frac{1}{3}\int \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

Let $u = x - \frac{1}{2}$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{(u+\frac{1}{2})-2}{u^2+3/4}\,du$$

$\frac{du}{dv} = 1$

$du = dx$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{u}{u^2+3/4}\,du + \frac{1}{3}\int \frac{+3/2}{u^2+3/4}\,du$$

So $x = u + \frac{1}{2}$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{\cancel{u}}{w}\cdot\frac{dw}{2\cancel{u}} + \frac{1}{3}\cdot\frac{+3}{2}\int \frac{1}{u^2+3/4}\,du$$

Let $w = u^2 + 3/4$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|w| + \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\arctan\left(\frac{u}{\sqrt{3}/2}\right) + C$$

$\frac{dw}{du} = 2u$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln\left|u^2+\tfrac{3}{4}\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C$$

$\frac{dw}{2u} = du$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$