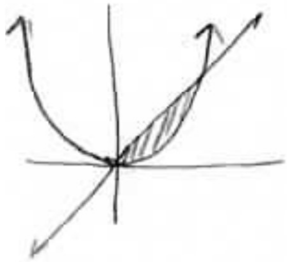
Exam 2b **Calculus 2** **3/1/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Set up an integral for the area of the region bounded between $y = x^2$ and $y = 3x$.



$x^2 = 3x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$

$$\boxed{\int_0^3 3x - x^2 \, dx}$$

Great

2. Set up an integral for the volume of the solid obtained when the region from #1 is rotated around the x-axis.

$$\int_0^3 \pi(3x)^2 - \pi(x^2)^2 \, dx$$

Great

3. If the work required to stretch a spring 1 ft beyond its natural length is 20 ft-lb, how much work (in ft-lb) is needed to stretch it 6 in. beyond its natural length?

$$\int_0^1 k\cdot x\,dx = 20$$

$$k\cdot \frac{x^2}{2}\Big|_0^1 = 20$$

$$\frac{1}{2}k = 20$$

$$k = 40 \ \frac{lbs}{ft}$$

$$\text{Work} = \int_0^{1/2} 40x\,dx$$

$$= 20x^2\Big|_0^{1/2}$$

$$= 20\cdot\left(\tfrac{1}{2}\right)^2$$

$$= 5 \text{ ft-lbs}$$

4. Set up an integral for the length of the portion of the curve $y = \sqrt{x}$ from $(0, 0)$ to $(9, 3)$.

$$\text{Arc length} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$y = \sqrt{x}$$
$$y = (x)^{1/2}$$
$$y' = \frac{1}{2}(x)^{-1/2}$$
$$y' = \frac{1}{2\sqrt{x}}$$

$$= \int_0^9 \sqrt{1+\left[\frac{1}{2\sqrt{x}}\right]^2}\,dx$$

Great

5. Write integrals for the present and future values of an income stream of 3000 dollars a year, for a period of 5 years, if the continuous interest rate is 4 percent.

$$P.V. = \int_0^M P(t)\cdot e^{-rt}\,dt \qquad F.V. = \int_0^M P(t)\cdot e^{r(M-t)}\,dt$$

$$P.V. = \int_0^5 3000\cdot e^{-.04t}\,dt \qquad F.V. = \int_0^5 3000\cdot e^{.04(5-t)}\,dt$$

Great

6. Find the x-coordinate of the center of mass of the region lying underneath the graph of the function $f(x) = \sqrt{x}$ over the interval $[0, 16]$.

$$\bar{x} = \frac{\int_0^{16} x\cdot\sqrt{x}\,dx}{\int_0^{16}\sqrt{x}\,dx}$$

$$= \frac{\frac{2}{5}x^{5/2}\Big|_0^{16}}{\frac{2}{3}x^{3/2}\Big|_0^{16}}$$

$$= \left(\frac{2}{5}\cdot 4^5\right) \div \left(\frac{2}{3}\ 4^3\right)$$

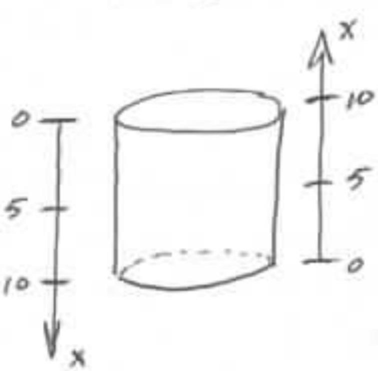
$$= \frac{2}{5}\cdot\frac{3}{2}\cdot 4^2$$

$$= \frac{48}{5}$$

7. Star is a calculus student at Enormous State University, and they're having some trouble. Star says "Geez! Calc 2 is so different from Calc 1! It used to be there was, like, just one right way to do things, right? But now I did this work problem as $\int_5^{10} 250\pi(10-x)\,dx$ but the professor was saying to do it $\int_0^5 250\pi x\,dx$. The thing is, they both worked out to the same answer! Is it just a coincidence?"

Help Star out. Explain to them as clearly as possible why both ways work for the same problem, or why you know it's just a coincidence.

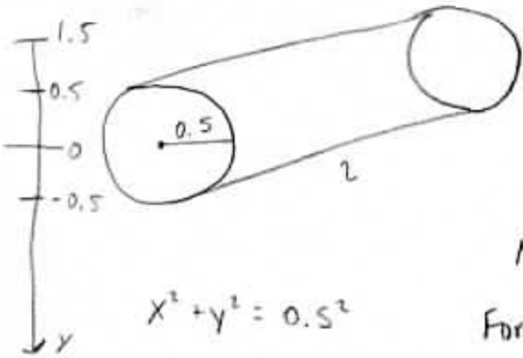
That's definitely tricky, but yes those are both good ways to do the problem. The difference is because there are some choices along the way where both ways work, but they look different. The choice you're seeing is about where you set up your axis. If you look at this picture, if you use the axis on the left then the top half of the tank goes from the 0 tick mark to the 5 tick mark, and the distance the stuff at x has to go to get out is just x, like the stuff at the 2 tick mark has 2 steps to go.



But if we use the axis on the right instead, then the top half of the tank goes with x values between 5 and 10, and the distance something at the 8 tick mark has to go to get out is $10-8=2$ steps. Then the axis on the right matches the integral $\int_5^{10} 250\pi(10-x)dx$, but the axis on the left matches the integral $\int_0^5 250\pi x\,dx$, and they have to be the same.

p.s. If you do the u-substitution $u=10-x$, it turns that first integral into the other 🙂

8. A gas station stores its gasoline in a tank under the ground. The tank is a cylinder lying horizontally on its side. (In other words, the tank is not standing vertically on one of its flat ends.) If the radius of the cylinder is 0.5 meter, its length is 2 meters, and its top is 1 meter under the ground, set up an integral for the total amount of work needed to pump the gasoline out of the tank to ground level. (The density of gasoline is 673 kilograms per cubic meter; use $g = 9.8m/s^2$).



Width : $2\sqrt{0.5^2 - y^2}$ m

Area : $2\sqrt{0.5^2 - y^2}$ m $\cdot$ 2 m

Volume : $4\sqrt{0.5^2 - y^2}$ $m^2 \cdot \Delta y$ m

Mass : $4\sqrt{0.5^2 - y^2} \cdot \Delta y$ $m^3 \cdot 673$ kg/m^3

Force : $2692\sqrt{0.5^2 - y^2} \cdot \Delta y \cdot 9.8$ N

Work : $26381.6\sqrt{0.5^2 - y^2} \cdot \Delta y \cdot (1.5 - y)$ J

$x^2 + y^2 = 0.5^2$

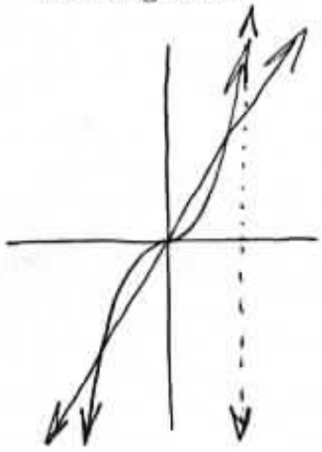
$\sqrt{x^2} = \sqrt{0.5^2 - y^2}$

$x = \sqrt{0.5^2 - y^2}$

$$\boxed{\text{Total Work: } \int_{-0.5}^{0.5} 26381.6(1.5 - y)\sqrt{0.5^2 - y^2}\, dy \text{ J}}$$

Excellent!

9. Consider the entire region bounded between $y = x^3$ and $y = 3x$. If this region is revolved around the axis $x = 2$, set up an integral or integrals for the volume of the resulting solid.



$$x^3 = 3x$$
$$x^3 - 3x = 0$$
$$x(x^2 - 3) = 0$$
$$x = 0 \text{ or } x = \pm\sqrt{3}$$

Shells!

$$\text{Volume} = \int_{-\sqrt{3}}^{0} 2\pi(2-x)(x^3-3x)\,dx$$
$$+ \int_{0}^{\sqrt{3}} 2\pi(2-x)(3x-x^3)\,dx$$

10. Consider a sphere with radius r. Treating it as a solid of revolution obtained from rotating half of a circle centered at the origin around one of the coordinate axes, show that the surface area of the sphere is $4\pi r^2$.

$$x^2 + y^2 = r^2$$

$$\sqrt{y^2} = \sqrt{r^2 - x^2}$$

$$y = \sqrt{r^2 - x^2}$$

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$f(u) = \sqrt{u} \quad u^{1/2} \qquad u = r^2 - x^2$$

$$f'(u) = \frac{1}{2} u^{-\frac{1}{2}} \qquad u' = -2x$$

$$-\cancel{2}x\left(\frac{1}{\cancel{2}}(r^2 - x^2)^{-1/2}\right)$$

$$\frac{-x}{\sqrt{r^2 - x^2}}$$

$$SA = \int_a^b 2\pi \left[f(x)\right] \sqrt{1 + [f'(x)]^2}\, dx$$

for other half

$$2\int_0^r 2\pi \left[\sqrt{r^2 - x^2}\right] \sqrt{1 + \left[\frac{-x}{\sqrt{r^2 - x^2}}\right]^2}\, dx$$

$$4\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$4\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{\frac{r^2 \cancel{- x^2} \cancel{+ x^2}}{r^2 - x^2}}\, dx$$

$$4\pi \int_0^r \cancel{\sqrt{r^2 - x^2}} \frac{\sqrt{r^2}}{\cancel{\sqrt{r^2 - x^2}}}\, dx$$

Nice!

$$4\pi \int_0^r r\, dx$$

$$4\pi \cdot rx$$

$$4\pi r(r) - \cancel{4\pi r(0)}$$

$$\boxed{4\pi r^2}$$