Exam 3a **Calculus 2** **3/29/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Determine the exact sum of the geometric series $4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

$a = 4$ $\quad 4 \cdot r^1 = -2 \quad S = \frac{a}{1-r} \quad S = \frac{4}{1-(-\frac{1}{2})} = \frac{8}{3}$

$r = -\frac{1}{2} \quad r^1 = -\frac{1}{2}$

Good

2. Find the first 3 partial sums of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!}$

$S_1 = \frac{(-1)^{(1)+1}}{(2(1)-1)!} = \frac{(-1)^2}{1!} = \frac{1}{1!} = 1$

$S_2 = \frac{(-1)^{(2)+1}}{(2(2)-1)!} = \frac{-1^3}{4-1} = \frac{-1}{3!} = -\frac{1}{6} + 1 = \frac{5}{6}$

$S_3 = \frac{(-1)^{(3)+1}}{(2(3)-1)!} = \frac{-1^4}{6-1} = \frac{1}{5!} = \frac{1}{120} + \frac{5}{6} = \frac{101}{120}$

Excellent!

3. Find the 5th degree MacLaurin polynomial for $f(x) = e^x$.

$f(x) = e^x$ $\quad$ $f(0) = e^0 = 1$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$

$f''(x) = e^x$ $\quad$ $f''(0) = 1$

$f'''(x) = e^x$ $\quad$ $f'''(0) = 1$

$f^{(4)}(x) = e^x$ $\quad$ $f^{(4)}(0) = 1$

$f^{(5)}(x) = e^x$ $\quad$ $f^{(5)}(0) = 1$

Excellent!

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

4. Determine whether the series $\sum_{n=2} \frac{3}{n\ln(n)}$ converges or diverges.

10

try integral test

positive ✓
continuous ✓
decreasing ✓

we know it is decreasing because of n's on denom and static # numerator

$$3\int_2^{\infty} \frac{1}{x\ln x}\,dx$$

u sub

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$$= \lim_{b\to\infty} 3\int_2^b \frac{1}{x \cdot u}\,x\,du$$

Well done!

$$= 3\lim_{b\to\infty} \ln(u)$$

$$= 3\lim_{b\to\infty} \ln(\ln(x)) \Big|_2^b$$

$$= 3\lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2))$$

irrelevant

diverges

According to the integral test, since $\int_2^{\infty} \frac{3}{x\ln x}\,dx$ diverges, then $\sum_{n=2}^{\infty} \frac{3}{n\ln n}$ also diverges because $f(x)$ is positive, continuous, and decreasing.

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n^2}$ converges or diverges.

$$\frac{1}{1^2} + \frac{3}{2^2} + \frac{1}{3^2} + \frac{3}{4^2} + \frac{1}{5^2} + \frac{3}{6^2} + \ldots$$

These terms are always less than $\frac{3}{n^2}$, and

$\sum \frac{1}{n^2}$ is a p-series with $p = 2$, so it converges, so $\sum \frac{3}{n^2}$ must also converge, so by the Comparison Test $\sum \frac{2+(-1)^n}{n^2}$ converges too.

10

6. Write the first 3 non-zero terms of the Taylor Series for $f(x) = \cos x$ centered at $x = \frac{\pi}{2}$.

$f(x) = \cos x \quad f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin x \quad f'(\frac{\pi}{2}) = -1$

$f''(x) = -\cos x \quad f''(\frac{\pi}{2}) = 0$

$f'''(x) = \sin x \quad f'''(\frac{\pi}{2}) = 1$

$f''''(x) = \cos x \quad f''''(\frac{\pi}{2}) = 0$

$f^{(5)}(x) = -\sin x \quad f^{(5)}(\frac{\pi}{2}) = -1$

$$0 - 1x + \frac{0x^2}{2!} + \frac{x^3}{3!} + \frac{0x^4}{4!} - \frac{x^5}{5!}$$

$\Downarrow$ get rid of zeroes

$$-x + \frac{x^3}{3!} - \frac{x^5}{5!}$$

$\Downarrow$ plug in $-\frac{\pi}{2}$

$$-\left(x - \frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} - \frac{\left(x - \frac{\pi}{2}\right)^5}{5!}$$

Excellent!

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, this series stuff is crazy. So we were supposed to say if $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges or diverges, right? So I said it converges by comparison to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, which was totally one of the choices on the multiple choice, right? But they said it was wrong! They're like practically the same thing, so what could be wrong about it?"

Help Biff out by explaining the validity of his conclusion. You do not need to say whether the series converges or not; you're just commenting on the proposed reasoning.

By the comparison Test: $a_n \le b_n$ with $\sum b_n$ convergent, then $\sum a_n$ converges also. So $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is converges by p-series

then we set $b_n = \frac{1}{n^2}$, $a_n = \frac{1}{n^2-1}$

Because ~~$\ge$~~ $n^2-1 \le n^2$, then $\frac{1}{n^2-1} \ge \frac{1}{n^2}$

So $a_n \ge b_n$ by the comparison Test. $a_n \ge b_n$ with $\sum b_n$ divergent, then $\sum a_n$ diverges also. ~~But~~ On there, $a_n \ge b_n$ but b_n is not divergent other. $\sum b_n$ converges, but $a_n \le b_n$. So you can't use the comparision test to do it.

Great

8. Use a Maclaurin polynomial with at least 4 terms to approximate $e^{0.1}$.

by Maclaurin polynomial : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n$

We know $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

So $e^{0.1} \approx 1 + \cancel{0} 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3 \times 2}$

$\approx 1.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6}$

$\approx 1.1 + 0.005 + 0.00016667$

≈ 1.10516667

Excellent!

9. Use a Maclaurin polynomial of at least 8^{th} degree to approximate $\int_0^1 \cos(x^2)\,dx$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$\cos(x^2) \approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}$$

$$\int_0^1 \cos(x^2)\,dx \approx \int_0^1 \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}\right)dx$$

$$= \left[x - \frac{x^5}{5\cdot 2} + \frac{x^9}{9\cdot 24} - \frac{x^{13}}{13\cdot 720} + \frac{x^{17}}{17\cdot 40320}\right]_0^1$$

$$\left((1) - \frac{(1)^5}{10} + \frac{(1)^9}{216} - \frac{(1)^{13}}{9360} + \frac{(1)^{17}}{685440}\right)$$

$.9045242509$

Excellent!

10. For which values of x does $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n$ converge?

Try Rat. Test!

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{[2(n+1)+1]!}}{\frac{(-1)^n x^n}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{1x^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{1x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(2n+1)!} \cdot x}{(2n+3)(2n+2)\cancel{(2n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{(2n+3)(2n+2)} \right|$$

$$= 0$$

So since $0 < 1$ no matter what x is, this series converges no matter what x is