Exam 3b **Calculus 2** **3/29/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Determine the exact sum of the geometric series $4-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

$\left|-\frac{1}{2}\right|<1$

$$\frac{4}{1-\left(-\frac{1}{2}\right)}=\boxed{\frac{8}{3}}$$

Good

2. Find the first 3 partial sums of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)!}$

$\frac{1}{1!}$, $-\frac{1}{3!}$, $\frac{1}{5!}$

$-\frac{1}{6}$

$$\boxed{\begin{aligned} S_1 &= 1 \\ S_2 &= 1-\frac{1}{6}=\frac{5}{6} \\ S_3 &= 1-\frac{1}{6}+\frac{1}{120}=\frac{101}{120} \end{aligned}}$$

Great

3. Find the 5th degree MacLaurin polynomial for $f(x) = e^x$.

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$f'''(x) = e^x$ $\quad f'''(0) = 1$

$f^{(4)}(x) = e^x$ $\quad f^{(4)}(0) = 1$

$f^{(5)}(x) = e^x$ $\quad f^{(5)}(0) = 1$

$$\boxed{P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \frac{x^5}{5!}}$$

Great!

4. Determine whether the series $\sum_{n=2}^{\infty} \frac{3}{n \ln(n)}$ converges or diverges.

0

try integral test

positive ✓

continuous ✓

decreasing ✓

we know it is decreasing because of n's on denominator and static # numerator

$$3\int_2^\infty \frac{1}{x \ln x}\, dx$$

u sub

$u = \ln x$

$du = \frac{1}{x}\, dx$

$x\, du = dx$

$$= \lim_{b\to\infty} 3\int_2^b \frac{1}{x \cdot u}\, x\, du$$

Well done!

$$= 3\lim_{b\to\infty} \ln(u)$$

$$= 3\lim_{b\to\infty} \ln(\ln(x)) \Big|_2^b$$

$$= 3\lim_{b\to\infty} \ln(\ln(b)) - \underbrace{\ln(\ln(2))}_{\text{irrelevant}}$$

diverges

According to the integral test, since $\int_2^\infty \frac{3}{x \ln x}\, dx$ diverges, then $\sum_{n=2}^{\infty} \frac{3}{n \ln n}$ also diverges because $f(x)$ is positive, continuous, and decreasing.

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$ converges or diverges.

I know that $\frac{1}{\sqrt{n}}$ diverges because of a p-series with $p<1$

The numerator will always be either 1 or 3.

$1 \leq 2+(-1)^n$

$\frac{1}{\sqrt{n}} \leq \frac{2+(-1)^n}{\sqrt{n}}$ Since $\frac{2+(-1)^n}{\sqrt{n}}$ is greater than or equal to $\frac{1}{\sqrt{n}}$, the series also diverges according to the Comparison Test.

Excellent!

6. Write the first 3 non-zero terms of the Taylor Series for $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$.

$f(x) = \sin x \quad f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x \quad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \quad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \quad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{(4)}(x) = \sin x \quad f^{(4)}\left(\frac{\pi}{2}\right) = 1$

$$\boxed{P(x) = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}}$$

Great!

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, this series stuff is crazy. So we were supposed to say if $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges or diverges, right? So I said it converges by comparison to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, which was totally one of the choices on the multiple choice, right? But they said it was wrong! They're like practically the same thing, so what could be wrong about it?"

Help Biff out by explaining the validity of his conclusion. You do not need to say whether the series converges or not; you're just commenting on the proposed reasoning.

The reasoning is not valid, because $a_n \geq$ to b_n which means a_n and b_n must be divergent in order for the comparison test to be true, but in this problem $n^2-1 < n^2$ so by the comparison test you can not conclude that it converges or diverges, because $\frac{1}{n^2}$ is convergent but it is less than $\frac{1}{n^2-1}$ and the comparison test states "$a_n \geq b_n$ with Σb_n divergent then Σa_n diverges also." but the p-series test say $\frac{1}{n^2}$ converges because $p > 1$

Excellent!

8. Use a Maclaurin polynomial with at least 4 terms to approximate $e^{0.1}$.

I know $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

$$\boxed{e^{0.1} \approx 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24}}$$

$e^{0.1} \approx 1.10517$

Good

9. Use a Maclaurin polynomial of at least 8^{th} degree to approximate $\int_0^1 \cos(x^2)\,dx$

$f(x) = \cos x \quad f(0) = 1_0$

$f'(x) = -\sin x \quad f'(0) = 0_1 \qquad \cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$

$f''(x) = -\cos x \quad f''(0) = -1_2$

$f'''(x) = \sin x \quad f'''(0) = 0_3 \qquad \cos(x^2) \approx 1 - \frac{(x^2)^2}{2} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!}$

$4 = 1_4$

$5 = 0_5 \qquad \cos(x^2) \approx 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}$

$6 = -1_6$

$7 = 0_7$

$8 = 1_8$

$$\int_0^1 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}\,dx$$

$$x - \frac{x^5}{10} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \frac{x^{17}}{17\cdot 8!}\Big]_0^1$$

Excellent!

$$\boxed{\left(1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} + \frac{1}{685440}\right)}$$

10. For which values of x does $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n$ converge?

Try Rat. Test!

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{[2(n+1)+1]!}}{\frac{(-1)^n x^n}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{1x^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{1x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(2n+1)!} \cdot x}{(2n+3)(2n+2)\cancel{(2n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{(2n+3)(2n+2)} \right|$$

$$= 0$$

So since $0 < 1$ no matter what x is, this series converges no matter what x is